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Elasto-plastic models with continuously distributed dislocations and disclinations

Sanda CLEJA-TIGOIU[1,a]

University of Bucharest, Faculty of Mathematics and Computer Science, Romania

[a]stigoiu@yahoo.com, tigoiu@fmi.unibuc.ro

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Abstract. This paper deals with a mathematical model able to describe the presence of lattice defects of crystalline materials such as dislocations and disclinations. Within the constitutive framework of second order plasticity developed by the author, the evolution equations required to describe the disclinations compatible with the screw dislocations are derived.

Introduction

The plastic deformability of metals, which are crystalline materials, is produced by the lattice defects existing at the micro level, see Kröner [7]. To define the plastic part of the deformation (plastic distortion), $\mathbf{F}^p$, locally relaxed configurations (l.r.c.) are associated with each particle. These (l.r.c.) could not be fit together to restore a continuous body. Based on the affine connection, herein referred to as plastic connection and denoted by Γ^p, it becomes possible to introduce a geometry on the material structure which consists of the so-called plastically deformed configurations or configurations with torsion, see the constitutive framework of second order finite elasto-plasticity developed by Cleja-Ţigoiu [1, 2], in terms of the pair composed of the plastic distortion $\mathbf{F}^p$ and the plastic connection Γ^p.

In this paper, a peculiar mathematical problem is analyzed: find the disclinations compatible with an appropriate evolution equation in such a way that the micro balance equations are satisfied when the distribution of dislocations is given. To give the mathematical description of the problem, we provide the general constitutive framework which is able to capture the dislocations and disclinations. We mention here the direction developed by Clayton et al. [3] within a micropolar elasto-plastic model in order to emphasize *translational* (dislocation) and *rotational* (disclination) defects. In the paper by Fressengeas et al. [5], within the small deformation formalism, dislocations are generated not only by the plastic incompatibility, but also by the disclination mobility.

In the present paper, dislocations and disclinations are lattice defects of interest. The measure of *dislocations* is characterized either by the *non-zero curl of the plastic distortion*, which means that the plastic distortion can not be derived from a certain potential, or by a tensorial field, say the Noll dislocation density $\boldsymbol{\alpha}$, see Noll [8], which leads to the non-zero torsion of the plastic connection. Disclinations have been related to a certain second order tensor $\mathbf{\Lambda}$ which occurs in the expression of the plastic connection and generates a non-zero curvature, contrary de Wit [4] where a measure of the disclination is considered to be a second order curvature tensor. The response of the material is (second order) elastic with respect to the plastically deformed configuration, at whose level the presence of micro defects can be emphasized. The elasto-plastic behavior of the material is restricted to satisfy the imbalance free energy principle, following Gurtin [6] and Cleja-Ţigoiu [1], which is defined in terms of the free energy and internal power. The macro and micro forces satisfy their own balance laws. Here the free energy is assumed to depend on the elastic strain and various geometrical measures of the defects.

The following relationships, notations and definitions are used herein:
$\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$, $\mathbf{u}\otimes\mathbf{v}$ denote the scalar, cross and tensorial products of vectors, respectively; $\mathbf{a}\otimes\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}$ are defined to be the second and third order tensors, respectively, given by $(\mathbf{a}\otimes\mathbf{b})\mathbf{u}=\mathbf{a}(\mathbf{b}\cdot\mathbf{u})$ and $(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\mathbf{u}=(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\cdot\mathbf{u})$, for all vectors $\mathbf{u}\in\mathcal{V}$; the tensorial product $\mathbf{A}\otimes\mathbf{a}$ for $\mathbf{a}\in\mathcal{V}$ is a third order tensor with the following property $(\mathbf{A}\otimes\mathbf{a})\mathbf{v}=\mathbf{A}(\mathbf{a}\cdot\mathbf{v}), \forall\mathbf{v}\in\mathcal{V}$; $\mathbf{I}$ is the identity tensor in Lin, $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}\in Lin$; $\nabla\mathbf{A}$ is the gradient of the field $\mathbf{A}$, $\nabla\mathbf{A}=\dfrac{\partial A_{ij}}{\partial x^k}\mathbf{e}^i\otimes\mathbf{e}^j\otimes\mathbf{e}^k$; the *curl operator* is defined for any smooth second order field, say $\mathbf{A}$, by

$(\text{curl}\mathbf{A})(\mathbf{u}\times\mathbf{v})=(\nabla\mathbf{A})\mathbf{u})\mathbf{v}-((\nabla\mathbf{A})\mathbf{v})\mathbf{u}, \quad \forall \quad \mathbf{u},\mathbf{v}\in\mathcal{V}.$

$Lin(\mathcal{V},Lin)=\{\mathbf{N}:\mathcal{V}\longrightarrow Lin, \quad \text{linear}\}$ defines the space of all third order tensors, whose elements are given by $\mathbf{N}=N_{ijk}\mathbf{i}^i\otimes\mathbf{i}^j\otimes\mathbf{i}^k$. In a Cartesian system of coordinates, the scalar product of two second order tensors $\mathbf{A}$ and $\mathbf{B}$ is defined as $\mathbf{A}\cdot\mathbf{B}:=\text{tr}(\mathbf{A}\mathbf{B}^T)=A_{ij}B_{ij}$, while the scalar product of two third order tensors $\mathbf{N}$ and $\mathbf{M}$ is given by $\mathbf{N}\cdot\mathbf{M}=N_{ijk}M_{ijk}$.

The third order tensor field $\mathbf{\Gamma}[\mathbf{F}_1,\mathbf{F}_2]$ is generated by a third order field $\mathbf{\Gamma}$ together with the second order tensors $\mathbf{F}_1$ and $\mathbf{F}_2$ via the following formula: $(\mathbf{\Gamma}[\mathbf{F}_1,\mathbf{F}_2]\mathbf{u})\mathbf{v}=(\mathbf{\Gamma}(\mathbf{F}_1\mathbf{u}))\,\mathbf{F}_2\mathbf{v}$, $\forall\mathbf{u},\mathbf{v}\in\mathcal{V}$.
For any $\mathbf{\Lambda}_1\in Lin$ and $\mathbf{\Lambda}_2\in Lin$, we define the following associated third order tensor, denoted by $\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2$, $((\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\mathbf{u})\mathbf{v}=(\mathbf{\Lambda}_1\mathbf{u})\times(\mathbf{\Lambda}_2\mathbf{v}), \quad \forall \quad \mathbf{u},\mathbf{v}$.
For any third order tensor, $\mathcal{A}$, we define the vector field, $\text{tr}_{(2)}\mathcal{A}$, by the following relationship valid for all vectors $(\text{tr}_{(2)}\mathcal{A})\cdot\mathbf{u}=\text{tr}(\mathcal{A}\mathbf{u})$.
Two types of second order tensors, $\mathcal{A}\odot\mathcal{B}$ and $\mathcal{A}\;_r\odot\mathcal{B}$ will be associated with any pair of third order tensors $\mathcal{A},\mathcal{B}$, according to the following the rules valid for all $\mathbf{L}\in Lin$

$$\begin{array}{l}(\mathcal{A}\odot\mathcal{B})\cdot\mathbf{L}=\mathcal{A}[\mathbf{I},\mathbf{L}]\cdot\mathcal{B}=\mathcal{A}_{isk}L_{sn}\mathcal{B}_{ink}\\ (\mathcal{A}\;_r\odot\mathcal{B})\cdot\mathbf{L}=\mathcal{A}\cdot(\mathbf{L}\mathcal{B})=\mathcal{A}_{ijk}L_{in}\mathcal{B}_{njk}.\end{array} \tag{1}$$

3 Geometric relationships

Let $\mathbf{F}(\mathbf{X},t)=\nabla\chi(\mathbf{X},t)$ be the deformation gradient at time t, $\mathbf{X}\in\mathcal{B}$, and $\mathbf{\Gamma}=\mathbf{F}^{-1}\nabla\mathbf{F}$ be the motion connection or the material connection. $\nabla\mathbf{F}$ is a gradient in the reference configuration, while the gradient in the configuration with torsion $\mathcal{K}$, $\nabla_{\mathcal{K}}\mathbf{F}$, is calculated by $\nabla_{\mathcal{K}}\mathbf{F}:=(\nabla\mathbf{F})(\mathbf{F}^p)^{-1}$. **Ax.1** The decomposition of the second order deformation, $(\mathbf{F},\mathbf{\Gamma})$, associated with the motion of the body $\mathcal{B}$, into the elastic, $(\mathbf{F}^e,\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}})$, and the plastic second order deformations, $(\mathbf{F}^p,\overset{(p)}{\mathbf{\Gamma}})$, respectively, is given by

$$\mathbf{F}=\mathbf{F}^e\mathbf{F}^p, \quad \mathbf{\Gamma}=\overset{(p)}{\mathbf{\Gamma}}+(\mathbf{F}^p)^{-1}\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}[\mathbf{F}^p,\mathbf{F}^p], \quad \mathbf{\Gamma}=\mathbf{F}^{-1}\nabla\mathbf{F}. \tag{2}$$

Here, the plastic connection, $\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}$, with respect to the configuration with torsion $\mathcal{K}$ is related to the plastic connection, $\overset{(p)}{\mathbf{\Gamma}}$, previously defined with respect to the reference configuration, by

$$\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}=-\mathbf{F}^p\overset{(p)}{\mathbf{\Gamma}}[(\mathbf{F}^p)^{-1},(\mathbf{F}^p)^{-1}]. \tag{3}$$

The plastic metric tensor, $\mathbf{C}^p$, and strain gradient, $\mathbf{C}$, are defined with respect to the reference configuration, while the elastic metric tensor, $\mathbf{C}^e$, is defined in the configuration with torsion by

$$\mathbf{C}^p:=(\mathbf{F}^p)^T\mathbf{F}^p, \quad \mathbf{C}^e=(\mathbf{F}^p)^{-T}\mathbf{C}(\mathbf{F}^p)^{-1}, \quad \mathbf{C}=\mathbf{F}^T\mathbf{F}. \tag{4}$$

The Bilby-type plastic connection is defined in a coordinate system as

$$\overset{(p)}{\mathcal{A}}:=(\mathbf{F}^p)^{-1}\nabla\mathbf{F}^p \tag{5}$$

Ax.2 The plastic connection with respect to the reference configuration has a metric property with respect to the plastic metric tensor, $\mathbf{C}^p$. This means by definition that, for all vectors , the following equality holds:

$$(\nabla\, \mathbf{C}^p)\mathbf{u} = (\overset{(p)}{\mathbf{\Gamma}}\, \mathbf{u})^T \mathbf{C}^p + \mathbf{C}^p(\overset{(p)}{\mathbf{\Gamma}}\, \mathbf{u}),\ \forall\, \mathbf{u} \in \mathcal{V}. \tag{6}$$

We introduce the expression for the plastic connection with respect to the reference configuration developed by Cleja-Tigoiu [1], which has a metric property with respect to $\mathbf{C}^p$, and allows a representation in the following form

$$\overset{(p)}{\mathbf{\Gamma}} = \overset{(p)}{\mathcal{A}} + (\mathbf{C}^p)^{-1}(\mathbf{\Lambda} \times \mathbf{I}). \tag{7}$$

Here the third order tensor, $\mathbf{\Lambda} \times \mathbf{I}$, is generated by the second order tensors, $\mathbf{\Lambda}$ and $\mathbf{I}$. $\mathbf{\Lambda}$ is referred to as the *disclination* tensor.

The following results were proven in [2]:

1. The second order torsion tensor, $\mathcal{N}^p$, is associated with the Cartan torsion, $\mathbf{S}^p$, and is expressed by

$$\begin{aligned} &\mathcal{N}^p = (\mathbf{F}^p)^{-1}\text{curl}\mathbf{F}^p + (\mathbf{C}^p)^{-1}\big((\text{tr}\ \mathbf{\Lambda})\mathbf{I} - (\mathbf{\Lambda})^T\big), \\ &(\mathbf{S}^p\mathbf{u})\mathbf{v} := (\overset{(p)}{\mathbf{\Gamma}}\, \mathbf{u})\mathbf{v} - (\overset{(p)}{\mathbf{\Gamma}}\, \mathbf{v})\mathbf{u} \quad = \mathcal{N}^p(\mathbf{u} \times \mathbf{v}),\ \forall\, \mathbf{v}, \mathbf{u} \in \mathcal{V}. \end{aligned} \tag{8}$$

2. The *Riemannian curvature tensor*, $\mathcal{R}$, belonging to the connection $\mathbf{\Gamma}$ and the curvature tensor, $\mathcal{R}^\Lambda$, formed from the third order tensor ($\mathbf{\Lambda} \times \mathbf{I}$) are characterized in a system of coordinates by

$$\begin{aligned} &(\mathcal{R}\mathbf{u})\mathbf{v} = ((\nabla\mathbf{\Gamma})\mathbf{u})\mathbf{v} - ((\nabla\mathbf{\Gamma}\mathbf{v})\mathbf{u} + (\mathbf{\Gamma}\mathbf{u})\mathbf{\Gamma}\mathbf{v} - (\mathbf{\Gamma}\mathbf{v})\mathbf{\Gamma}\mathbf{u}, \\ &((\mathcal{R}^\Lambda\mathbf{u})\mathbf{v})\mathbf{w} \cdot \mathbf{z} = \mathbf{r}^\Lambda(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}), \quad \text{with } \mathbf{r}^\Lambda = \text{curl}\mathbf{\Lambda} + (\text{Adj}\mathbf{\Lambda})^T. \end{aligned} \tag{9}$$

Here $Adj(A)$ is associated with $A \in Lin$ in such that the following equality holds: $(\mathbf{u}\times\mathbf{v})\cdot Adj(A)\mathbf{w} = (A\mathbf{u} \times A\mathbf{v}) \cdot \mathbf{w}$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$.

Remark Formula (8) can be regarded as an extension to the finite deformation model of the equation defining the incompatibility of the plastic strain associated with the dislocation density tensor, α, (see de Wit [4] and Fressengeas et al. [5])

$$\begin{aligned} &\boldsymbol{\alpha} = \text{curl}\boldsymbol{\varepsilon}^p + \big((\text{tr}\ \boldsymbol{\kappa}^p)\mathbf{I} - (\boldsymbol{\kappa})^T\big), \quad \text{where} \\ &\boldsymbol{\alpha} = \text{curl}(\mathbf{U}^p), \quad \nabla\mathbf{u} = (\mathbf{U}^p) + (\mathbf{U}^e), \quad \boldsymbol{\varepsilon}^p := \frac{1}{2}(\mathbf{U}^p + (\mathbf{U}^p)^T). \end{aligned} \tag{10}$$

In the finite deformation model, the dislocation density tensor is defined as $\boldsymbol{\alpha} = (\mathbf{F}^p)^{-1}\text{curl}\mathbf{F}^p$ which occurs in formula (8).

Constitutive elasto-plastic model

For the sake of simplicity, herein we restrict ourselves to a version of the general model for elasto-pastic materials with structural defects, such as dislocations and dislinations, that can be found in Cleja-Ţigoiu [2].

$$\begin{aligned} &\frac{1}{\rho}\mathbf{T} = 2\mathbf{F}(\partial_{\mathbf{C}}\psi)\mathbf{F}^T \text{elastic type constitutive equation:} \\ &\text{div}\ \mathbf{T} + \rho\mathbf{b} = 0 \quad \text{balance equation for macro stress;} \\ &J^p\ \mathbf{\Upsilon}^\lambda = \text{div}\ \big(J^p\ \boldsymbol{\mu}^\lambda(\mathbf{F}^p)^{-T}\big) \quad \text{micro balance equation;} \\ &\psi = \psi^e(\mathbf{C}^e) + \psi^d(\mathbf{\Lambda}, \nabla\mathbf{\Lambda}), \quad \psi^d := \frac{\kappa_2}{2}\beta_2^2\nabla\mathbf{\Lambda} \cdot \nabla\mathbf{\Lambda} \text{free energy density;} \\ &\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda = \partial_{\nabla\mathbf{\Lambda}}\psi^d(\nabla\mathbf{\Lambda}.) \end{aligned} \tag{11}$$

Various Mandel-type stress measures are associated with appropriate stresses as follows:

$$\frac{1}{\rho_0}\mathbf{\Sigma}_0^p = \frac{1}{\tilde{\rho}}(\mathbf{F}^p)^T\mathbf{\Upsilon}^p(\mathbf{F}^p)^{-T}, \quad \frac{1}{\rho_0}\mathbf{\Sigma}_0 = \frac{1}{\rho}\mathbf{F}^T\mathbf{T}\mathbf{F}^{-T}, \quad \frac{1}{\rho_0}\mathbf{\Sigma}_0^\lambda = \frac{1}{\tilde{\rho}}(\mathbf{F}^p)^T\mathbf{\Upsilon}^\lambda(\mathbf{F}^p)^{-T}. \tag{12}$$

Viscoplastic type dissipative evolution equations are postulated for the plastic distortion and disclination:

$$\begin{aligned} \xi_3\, \dot{\mathbf{\Lambda}} &= \frac{1}{\rho_0}\mathbf{\Sigma}_0{}^\lambda + \big(\overset{(p)}{\mathcal{A}} \odot \frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda\big) - \big(\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda \,{}_r\!\odot \overset{(p)}{\mathcal{A}}\big) - \frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda\big(\mathrm{tr}_{(2)}(\overset{(p)}{\mathcal{A}})\big), \\ \xi_1\, \mathbf{l}^p &= \frac{1}{\rho_0}(\mathbf{\Sigma}_0 - \mathbf{\Sigma}_0{}^p) + (\mathbf{F}^p)^T\partial_{\mathbf{F}^p}\psi \quad \text{where} \quad \mathbf{l}^p = -(\mathbf{F}^p)^{-1}\dot{\mathbf{F}}^p, \end{aligned} \tag{13}$$

such that they are compatible with an appropriate dissipation inequality:

$$\xi_1\, \mathbf{l}^p \cdot \mathbf{l}^p + \xi_3\, \dot{\mathbf{\Lambda}} \cdot \dot{\mathbf{\Lambda}} \geq 0. \tag{14}$$

The following notations have been used in the previous formulae: β_2 is a length scale parameter; the micro forces, $\mathbf{\Upsilon}^\lambda$, and the micro momentum, $\boldsymbol{\mu}^\lambda$, are associated with disclinations, while the relationship between the micro stress momenta is given by

$$\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda := (\mathbf{F}^p)^{-T}\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda[(\mathbf{F}^p)^T, (\mathbf{F}^p)^T]. \tag{15}$$

Here $\tilde{\rho}$ and ρ_0 are the mass densities in the configuration with torsion and in the reference configuration, respectively.

Disclination generated by a plastic distortion

The Burgers vector can be defined in terms of the plastic distortion, $\mathbf{F}^p$, by considering a closed curve (*circuit*), C_0, in the reference configuration and a surface, $\mathcal{A}_0$, with the normal, $\mathbf{N}$, bounded by C_0,

$$\mathbf{b} = \int_{C_0} \mathbf{F}^p\, d\mathbf{X} = \int_{\mathcal{A}_0} (\mathrm{curl}(\mathbf{F}^p))\mathbf{N} dA = \int_{\mathcal{A}_\mathcal{K}} \boldsymbol{\alpha}_\mathcal{K}\mathbf{n}_\mathcal{K} dA_\mathcal{K}. \tag{16}$$

Noll's dislocation density, $\boldsymbol{\alpha}_\mathcal{K}$, and Burgers vector, $\mathbf{b}$, are given by $\boldsymbol{\alpha}_\mathcal{K} \equiv \frac{1}{\det\mathbf{F}^p}(\mathrm{curl}(\mathbf{F}^p))(\mathbf{F}^p)^T$ and $\mathbf{b} \simeq \mathrm{curl}(\mathbf{F}^p)\mathbf{N}\ \mathrm{area}(\mathcal{A}_0)$. In crystal plasticity, the presence of defects inside crystals is measured by a non-vanishing Burgers vector. The integral representation (16) shows that a non-zero *curl* of plastic distortion, supposed to be continuum and non-zero in a certain material neighborhood, leads to a non-vanishing Burgers vector.

Definition We say that the plastic distortion, $\mathbf{F}^p$, characterizes *a screw dislocation* if the generated Burgers vector through a circuit with the appropriate normal, $\mathbf{N}$, is collinear to the normal, i.e. $\mathbf{b} \parallel \mathbf{N}$ in contrast to the *edge dislocation* for which $\mathbf{b} \perp \mathbf{N}$.

Let us introduce a Cartesian basis $(\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3)$. We denote by $\mathbf{b}$ the Burgers vector and consider various plastic distortions, $\mathbf{F}^p$, defined by the following formulae

(1) $\mathbf{F}^p = \mathbf{I} + \mathbf{e}_3 \otimes \boldsymbol{\tau}, \quad \boldsymbol{\tau} \perp \mathbf{e}_3,$ screw dislocation with $\mathbf{b} \parallel \mathbf{e}_3$,

(2) $\mathbf{F}^p = \mathbf{I} + \gamma\mathbf{e}_1 \otimes \mathbf{e}_3, \quad \mathbf{e}_1 \parallel \mathbf{b}$ edge dislocation,

(3) $\mathbf{F}^p = \mathbf{I} + \gamma\mathbf{e}_1 \otimes \mathbf{e}_3 + \nu\mathbf{e}_3 \otimes \mathbf{e}_3,$ a non-Schmid plastic flow, which means that $\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \frac{1}{\nu}(\dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_3 + \dot{\nu}\mathbf{e}_3 \otimes \mathbf{e}_3).$

In the above formulae, the shear and normal plastic rates are given by

$$\boldsymbol{\tau} : \mathcal{D} \subset R^2 \longrightarrow \mathcal{V}, \quad \gamma, \nu : \mathcal{D} \subset R^2 \longrightarrow R. \tag{17}$$

The dislocation density tensor can be represented in terms of a certain *edge* (which corresponds to $\rho_\perp \neq 0$) and *screw* (when $\rho_\odot \neq 0$) dislocations

$$\boldsymbol{\alpha} := \rho_\perp \mathbf{b} \otimes \boldsymbol{\xi} + \rho_\odot \mathbf{b} \otimes \mathbf{b}, \quad \mathbf{b} \cdot \boldsymbol{\xi} \neq 0. \tag{18}$$

Remark The Bilby connection can be calculated for any plastic distortions, however the expression of the plastic connection (5) requires the disclination tensor, Λ. Let us introduce the disclination tensor, $\mathbf{\Lambda}$, represented in terms of the Frank vector, $\boldsymbol{\omega}$, namely

$$\mathbf{\Lambda} := \eta \boldsymbol{\omega} \otimes \boldsymbol{\zeta}, \tag{19}$$

where $\boldsymbol{\zeta}$ is the tangent vector line for the disclination field and the scalar valued function, η, needs to be defined. If one takes constant values for $\boldsymbol{\zeta}$ and $\boldsymbol{\omega}$, it follows that

$$\nabla \mathbf{\Lambda} := \boldsymbol{\omega} \otimes \boldsymbol{\zeta} \otimes \nabla \eta, \quad \dot{\mathbf{\Lambda}} := \dot{\eta} \boldsymbol{\omega} \otimes \boldsymbol{\zeta}. \tag{20}$$

Hypothesis. We assume that Frank and Burgers vectors are orthogonal, $\boldsymbol{\omega} \cdot \mathbf{b} = 0$. This hypothesis corresponds to the physical meaning assigned to these types of lattice defects, see for instance Clayton et al. [3].

Within the proposed framework, we formulate the *problem*: *For a given plastic deformation process, find the disclination field as the solution of the balance equation for micro forces if dislocations are the sources of the evolution equation for the disclination.*

Let us exemplify how the solution of this problem can be found for case (1), where $\tau = \tau(x_1, x_2) \in \mathcal{V}$. A solution to the problem concerning the existence of a disclination field that is compatible with the dislocations generated by the plastic distortion introduced by (2) can be found in Cleja-Ţigoiu [2].

We emphasize the special issues related to this problem:

- Bilby's type plastic connection is expressed in the following form

$$\overset{(p)}{\mathcal{A}} := (\mathbf{F}^p)^{-1} \nabla \mathbf{F}^p = \mathbf{e}_3 \otimes \nabla \boldsymbol{\tau}, \quad \mathrm{tr}_{(2)} \overset{(p)}{\mathbf{A}} \cdot \mathbf{u} = \mathrm{tr}((\mathbf{e}_3 \otimes \nabla \boldsymbol{\tau}) \mathbf{u}) = 0; \tag{21}$$

- Consequently, the micro stress can be evaluated from (11) and it is found that

$$\begin{aligned} \boldsymbol{\Upsilon}^d &= \mathrm{div}\,\big(\boldsymbol{\mu}^\lambda (\mathbf{F}^p)^{-T}\big) = \kappa_2 \beta_2^2 \rho_0\, \Delta\eta \{\boldsymbol{\omega} \otimes \boldsymbol{\zeta} + (\boldsymbol{\zeta} \cdot \mathbf{e}_3)\, \boldsymbol{\omega} \otimes \boldsymbol{\tau}\} + \\ &+ \kappa_2 \beta_2^2 \rho_0 (\boldsymbol{\zeta} \cdot \mathbf{e}_3)\, \big(\boldsymbol{\omega} \otimes ((\nabla\, \boldsymbol{\tau}) \nabla \eta)\big) \end{aligned} \tag{22}$$

  Here $\Delta\eta$ denotes the Laplacian of the scalar function η.

- In the evolution equation for $\mathbf{\Lambda}$, the following expressions should be used:

$$\begin{aligned} &\overset{(p)}{\mathcal{A}} \odot \frac{1}{\rho_0} \boldsymbol{\mu}_0^\lambda = \kappa_2 \beta_2^2\, \big(\mathbf{e}_3 \otimes \nabla \boldsymbol{\tau}\big) \odot \big(\boldsymbol{\omega} \otimes \boldsymbol{\zeta} \otimes \nabla \eta\big) = 0, \\ &\frac{1}{\rho_0} \boldsymbol{\mu}_0^\lambda \,{}_r\!\odot \overset{(p)}{\mathcal{A}} = \kappa_2 \beta_2^2\, \big(\boldsymbol{\zeta} \cdot \big(\nabla \boldsymbol{\tau}\big) \nabla \eta\big)\, \mathbf{e}_3 \otimes \boldsymbol{\omega}. \end{aligned} \tag{23}$$

- The expression for the appropriate Mandel stress can be evaluated from (12), together with (22)

$$\begin{aligned} &\frac{1}{\rho_0} \boldsymbol{\Sigma}_0^\lambda = \kappa_2 \beta_2^2\, \{\Delta\eta \big(\boldsymbol{\omega} \otimes \boldsymbol{\zeta} + (\boldsymbol{\zeta} \cdot \mathbf{e}_3) \boldsymbol{\omega} \otimes \boldsymbol{\tau}\big) + \\ &+ (\boldsymbol{\zeta} \cdot \mathbf{e}_3) \boldsymbol{\omega} \otimes ((\nabla \boldsymbol{\tau}) \nabla \eta)\} - \kappa_2 \beta_2^2\, \Delta\eta (\boldsymbol{\zeta} \cdot \boldsymbol{\tau})(\boldsymbol{\omega} \otimes \mathbf{e}_3) - \\ &- \kappa_2 \beta_2^2\, \Delta\eta (\boldsymbol{\zeta} \cdot \mathbf{e}_3) \{\Delta\eta \mid \boldsymbol{\tau} \mid^2 + ((\nabla \boldsymbol{\tau}) \nabla \eta) \cdot \boldsymbol{\tau}\} \boldsymbol{\omega} \otimes \mathbf{e}_3. \end{aligned} \tag{24}$$

- The evolution equation for $\mathbf{\Lambda}$ written in (13) becomes

$$\frac{1}{\rho_0} {\boldsymbol{\Sigma}_0}^\lambda - \kappa_2 \beta_2^2\, \big(\boldsymbol{\zeta} \cdot \big(\nabla \boldsymbol{\tau}\big) \nabla \eta\big)\, \mathbf{e}_3 \otimes \boldsymbol{\omega} = \xi_3\, \dot{\eta} \boldsymbol{\omega} \otimes \boldsymbol{\zeta}, \tag{25}$$

  where the first term is given by (24) since $\mathrm{tr}_{(2)} \overset{(p)}{\mathcal{A}} = 0$.

Conclusions.

As a consequence that follows from (25) and (24), and under the special assumption that the disclination line, $\boldsymbol{\zeta}$, is orthogonal to the Burgers vector, namely $\mathbf{e}_3 \cdot \boldsymbol{\zeta} = 0$, the evolution equation for the density of the disclination is characterized by the non-local equation for the scalar function η

$$\xi_3 \dot{\eta} = \kappa_2 \beta_2^2 \Delta \eta, \tag{26}$$

and the compatibility condition is reduced to $\boldsymbol{\zeta} \cdot \boldsymbol{\tau} = 0$.

The same type of analysis can be provided for a more complex plastic distortion as given in (3).

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