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Orientational anisotropy and strength-differential effect in orthotropic elasto-plastic materials

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A B S T R A C T

This paper is devoted to elasto-plastic orthotropic model within the multiplicative elasto-plasticity, which describes the change of the orthotropic axes, i.e., orientational anisotropy and the strength-differential effect, when the yield condition is pressure insensitive and dependent on the third invariant of the stress. The orthotropy directions are characterized by Euler angles within the constitutive framework with small elastic strains, large elastic rotations and large plastic distortions. The presence of the plastic spin makes possible the description of the orientational anisotropy. We make herein an attempt to develop a hardening model, which includes the kinematic hardening given by Armstrong and Frederick (1966) law adapted to orthotropic material and isotropic hardening, and complies with the experimentally observed plastic yield and flow behaviour reported by Verma et al. (2011) in tension–compression–tension and compression–tension–compression. By pushing away to the actual configuration the material response, the rate form of the model with the objective derivatives expressed via the elastic rotations is characterized by a differential system for the following unknowns: the Cauchy stress, plastic part of deformation, tensorial and scalar hardening variables and Euler angles. We present the rate elasto-plastic model with a plastic spin in the case of an in-plane rotation of the orthotropy direction, and a plane stress, respectively. In the plane stress, the equation for the rate of the strain in the normal direction is first derived and subsequently the modified expression for the plastic multiplier associated with an in-plane rate of the deformation becomes available. Numerical simulations for the homogeneous deformation process on the sheets and comparisons with experimental data make possible a selection among the plastic spins introduced in this paper, aiming at obtaining a good agreement with the experiments performed for an in-plane stress state by Kim and Yin (1997). When the shear deformation of the plate is numerically simulated, the stabilization of the orientational anisotropy occurs in the presence of the plastic spin, in contrast with the unreasonable behaviour produced in the absence of the plastic spin.

1. Introduction

In this paper the plastic spin and non-quadratic orthotropic yield function, insensitive to the pressure and dependent on the third invariant of the stress, are combined to describe the elasto-plastic hardening material with strength-differential effect. The effect of the so-called *strength differential effect* of some metals has been experimentally observed by Hosford (1993) and Verma et al. (2011) with the compressive strengths lower than the tensile strengths and reported by Kuroda

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(2003), Nixon et al. (2010), Kuwabara (2007) with the flow stress in uniaxial compression larger than that in uniaxial tension. This is the rationale that led Cazacu and Barlat (2004) to extend the Drucker isotropic criterion dependent on the third invariant of the stress to orthotropic materials with strength differential effect, see also Cazacu et al. (2006) and Cazacu et al. (2010). Kuwabara (2007) considers that the accurate knowledge of the strength differential effect and the Baushinger effect in sheet metals are the crucial in predictive calculations for metal forming processes. Kuwabara (2007) reviewed experimental data and techniques for measuring the anisotropic plastic behaviour of metal sheets and tubes under a variety of loadings. Experimental difficulties in the conventional tension–compression experiments are related with the buckling of thin sheet metals. To overcome these difficulties, special devices for preventing buckling where attached to the specimens and the adhesively laminated specimens were prepared with pieces of sheets cut from uniaxially pre-strained sheets, see Yoshida (2000) and Kuwabara (2007). We mention that papers by Yoshida (2000) and Yoshida et al. (2002) address the constitutive model of cyclic plasticity for materials which exhibit a sharp point and subsequent abrupt yield drop followed by the yield plateau. When the flow stress in tension is not identical to that in compression, an asymmetric yield surface occurs. Thus these criteria (Hill, 1948; Barlat et al., 2003) are symmetric, so can not predict the yield stress difference in tension and compression. However, Hill (1948) yield criterion under plane stress state which involves additional linear terms is used by Verma et al. (2011) to derive an asymmetric yield function.

The rationale for using *non-quadratic yield functions*, which are insensitive to the pressure, has been imposed by their flexibility in the fitting of the initial yield shape and experimental data and by the necessity to perform a better description of the material behaviour taking into account the *r*-value data, as well as the yield stress data, see Cazacu and Barlat (2004), Barlat et al. (2005) and Soare et al. (2008), or to capture possible different yield stresses in tensile and compressive tests, see Cazacu and Barlat (2004) and Verma et al. (2011).

Non-quadratic polynomial yield functions are introduced in conjunction with their convexity by Soare and Barlat (2010) (generalizing the linear transformation) and Soare et al. (2008) in stress component representations. One method used in the plasticity of metals to describe the initial yield criterion is based on linear transformations, see for instance (Barlat et al., 2003, 2005); Kim et al., 2007, and the comments made in Soare and Barlat (2010). One or two linear transformations, say $\mathbf{L}'$ and $\mathbf{L}''$, are used to capture the material anisotropy, which generally means the presence of eighteen material constants. We also mention that the yield function has been represented as being dependent on the components of the stress with respect to orthotropic directions, i.e., rolling, transversal and normal directions, which are kept constant during the deformation process, as for instance in Barlat and Lian (1989) and Banabic et al. (2003, 2005).

The non-quadratic yield function introduced by Barlat et al. (2003) is used in the generalized finite element formulation for a mixed hardening elasto-plastic material with a non-linear non-associated flow rule performed by Taherizadeh et al. (2011, 2010) in the small deformation framework. In Korkolis and Kyriakides (2008), the non-quadratic Hosford (1979, 's) and Karafillis and Boyce (1993) yield functions are used in order to compare the numerical simulations and hydroforming experiments on aluminum tubes. The non-quadratic yield criterion based on spectral decomposition of the symmetric fourth rank elastic tensor (like the elasticity tensor) on the basis of the eigenvectors, represented through second order tensors, namely based on Kelvin modes, is proposed by Desmorat and Marull (2011). This criterion is adapted to describe the tension–compression yielding asymmetry, however an incomplete elasto-plastic model is presented therein.

The *hardening behaviour* of metals was observed in the experimental data reported by Phillips and Liu (1972), Ikegami (1979), Kim and Yin (1997), Khan and Jackson (1999), Hahm and Kim (2008), etc. A constitutive viscoplastic model of cyclic plasticity proposed by Yoshida (2000) and Yoshida et al. (2002) is able to describe the complex behaviour exhibited by certain steel sheets during the reverse loading. These authors experimentally emphasized the transient Baushinger strain and the permanent softening, where the reverse work hardening rate is lower than that during a forward deformation described by some kinematic and hardening rules. The tensorial hardening variable, i.e., the so-called back stress, is described by an evolution equation of the type proposed by Chaboche and Rosselier (1983) and Chaboche (2008). In order to improve the numerical prediction of the model when comparison with experimental data is performed, the Armstrong and Frederick (1966) hardening rule with two material constants is used by Taherizadeh et al. (2011, 2010). In Chung et al. (2005), the combined isotropic-kinematic hardening rule (formulated on the modified equivalent plastic work principle during a unidimensional process) is applied to the non-quadratic anisotropic yield function proposed by Barlat et al. (2003) in the plane stress state. In order to account for the Baushinger effect, the transient behaviour and permanent softening an improvement of the combined isotropic-kinematic hardening model proposed by Chung et al. (2005), has been elaborated in Verma et al. (2011). The two function parameters, which characterize the hardening law and are actually the slopes of the back stress evolutions, have been determined from the experimental hardening curves in uniaxial tension–compression tests. The Chung and Park (accepted for publication) consistency condition of "the combined type isotropic-hardening law of anisotropic yield functions with the full isotropic hardening law under the monotonously proportional loading" becomes useful to simplify the description of physical phenomena with theoretical arguments, similar to those related to the uniaxial process in the direct and reversal directions, and find the material function parameters.

In the present paper we do not make speculations having in mind the evolution equation for hardening variables and the yield function representation only, but we analyze the material response under uniaxial tensile test, i.e., under a proportional deformation process, based on the solution of the differential system which describes the model.

Plastic anisotropy is produced by the texture and lattice classes which characterize the metal microstructure and implies the distorsion of the yield surface shape and its evolution and the Baushinger effect, which is produced by a residual stress distribution, i.e., the so-called back stress, see Boehler, 1983. Various approaches to describe the anisotropy can be found in

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the literature. One method is based on the assumption concerning the existence of a symmetry group which renders constitutive function invariants, i.e., expressed in terms of an appropriate list of structural tensorial or scalar invariants, see for instance in Boehler (1983), Dafalias (1985), Cleja-Ţigoiu and Soós (1989), Cleja-Ţigoiu and Soós (1990), Cleja-Ţigoiu (2000a,b) and Miehe (2002).

In this paper the group of orthotropy is assumed to be g_6 characterized by Liu (1982) and Ting (1996) as follows

$$\begin{aligned} &g_6 = \{\mathbf{Q} \in Ort | \mathbf{Q}\mathbf{n}_i = \mathbf{n}_i \quad \text{or} \quad \mathbf{Q}\mathbf{n}_i = -\mathbf{n}_i, \quad i = 1,2,3\} \quad \Longleftrightarrow \\ &g_6 = \{\mathbf{Q} \in Ort | \mathbf{Q}(\mathbf{n}_i \otimes \mathbf{n}_i) = \mathbf{n}_i \otimes \mathbf{n}_i, \quad i = 1,2\}, \end{aligned} \tag{1}$$

where $\{\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3}\}$ denotes the orthotropy directions, while *Ort* denotes the set of all orthogonal transformations.

Another method already mentioned is based on *linear transformations* which involve the orthotropy, although no direct reference to the symmetry group (1) has been made. The linear fourth-order tensors which characterize these linear transformations ought to be invariant with respect to the orthotropy group, see also Cazacu et al. (2010), i.e., $\mathbf{L}'(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T) = \mathbf{Q}\mathbf{L}'(\boldsymbol{\sigma})\mathbf{Q}^T$ for all $\mathbf{Q} \in g_6$ and for all symmetric second order tensors $\boldsymbol{\sigma}$. The representation of such fourth-order tensors is provided (with the symmetry of the pairs of indices) in our notations in Appendix (A4) and it depends on nine material parameters, see also the comments and representation in Cazacu et al. (2010). The components of the linear transformations, see for instance (Banabic et al., 2003; Barlat et al., 2005), are determined by taking into account the r-value data, as well as the yield stress data and using the Newton–Raphson iteration method. In order to increase the number of anisotropic coefficients, the *n* linear orthotropic like transformations have been considered, see (Barlat et al., 2005; Barlat et al., 2007). Let us remark that the representation of the operator $\mathbf{L}'$ given by (15) from Barlat et al., 2005 and B given by (6) from Kim et al. (2007) do not have the property of transforming a deviator into a deviator, but the representation (17) from Barlat et al., 2005 has this property, as well as the appropriate linear transformation derived in Cazacu et al. (2010). Lode's type formulae can be seen for instance in Kachanov (1974) and hold for deviator tensors only, are frequently used in the afore mentioned paper, and thus the tensorial fields have to be deviatoric. In (Barlat et al., 2005, 2007) the yield functions have been written as functions of the *n*-set of the principal values associated with the transformed stress tensor via appropriate transformations. Note that such kind of representations for orthotropic functions is more restrictive than those representations in terms of the scalar and tensorial invariants that follow as a consequence of the theorems proved by Liu (1982), Wang (1970), Boehler (1983). We remark that the proper vectors of the transformed tensors via the linear transformations are not the same and this fact raises a question related to the physical meaning of the algebraic operation with their proper values. In the above mentioned papers devoted to the yield criteria no references are generally made relative to the evolution law for plastic strain. We mention that in Barlat et al. (2005) and in Kim et al. (2007), when the plastic flow rule is associated with the appropriate yield surfaces in terms of the linear transformations, certain singularities arise (when the derivative are taken) for the stress or strain state that can not be a priori excluded.

- In this paper, we adopt the constitutive framework based on the multiplicative decomposition of the deformation gradient, $\mathbf{F}$, into its elastic, $\mathbf{E}$, and plastic, $\mathbf{P}$, components called distortions, respectively

$$\mathbf{F} = \mathbf{E}\mathbf{P}, \tag{2}$$

within the constitutive framework of elasto-plastic materials with relaxed configurations and internal state variables, which has been proposed by Cleja-Ţigoiu, 1990 and Cleja-Ţigoiu and Soós, 1990. In this paper, the elastic distortion $\mathbf{E}$ describes the local mapping from the *isoclinic configuration* to the deformed configuration. The plastic distortion $\mathbf{P}$ characterizes the local deformation from the reference configuration to the isoclinic configuration. In order to define correctly, on a physical basis, the elastic and plastic distortions we use the so-called isoclinic configuration introduced by Teodosiu (1970), Mandel (1972), Kratochvill (1971). The indetermination in choosing the local relaxed configuration, which is attached to the crystalline lattice, has been eliminated by assuming that, in these isoclinic configurations, the corresponding crystalline directions are parallel to each other.

- The fact that the plastic distortion leaves the crystalline structure unchanged leads to the physically motivated assumption that the material response is invariant with respect to the geometrical transformations which preserve the crystalline lattice symmetry. As we fixed a reference configuration, the criterion for choosing the local relaxed configuration, namely defining the isoclinic configuration, determines uniquely the set of such configurations, apart from the orthogonal maps contained in the material symmetry group. As a consequence of the material symmetry concept proposed by Cleja-Ţigoiu and Soós (1989), Cleja-Ţigoiu and Soós (1990), all constitutive and evolution functions written with respect to the isoclinic configuration have to be invariant with respect to the material symmetry group, see the Theorem 1, below. In the case considered here, the symmetry group characterizes the orthotropy and this is assumed to be g_6.
- The multiplicative decomposition of the deformation gradient has been introduced, for instance, by Kröner (1960), Lee (1969), Teodosiu (1970), Rice (1971), and so on.
- Following Mandel (1972), we assume that the elastic strains are small, while the elastic rotations are large, i.e.,

$$\mathbf{E} = \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e, \quad \mathbf{U}^e \simeq \mathbf{I} + \boldsymbol{\varepsilon}^e, \quad \|\boldsymbol{\varepsilon}^e\| \ll 1 \tag{3}$$

The elastic rotations, $\mathbf{R}^e$, characterizes the passage from the isoclinic configurations to the deformed configurations and these rotations are described in terms of Euler's angles denoted by φ, ψ, θ, see Cleja-Ţigoiu and Iancu (2011).

- The kinematical consequences of the above hypothesis follow from the relationship between the velocity gradient, $\mathbf{L}$, and the elastic distortion-rate tensor, $\mathbf{L}^e$, and plastic distorsion-rate tensor $\mathbf{L}^p$, as a direct result of the multiplicative decomposition (2), namely

$$\mathbf{L} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{L}^e + \mathbf{E}\mathbf{L}^p\mathbf{E}^{-1}, \quad \mathbf{L}^e \equiv \dot{\mathbf{E}}\mathbf{E}^{-1}, \quad \mathbf{L}^p \equiv \dot{\mathbf{P}}\mathbf{P}^{-1}. \tag{4}$$

If one takes the symmetric and skew-symmetric parts of (4), then the following formulae yield

$$\begin{aligned} \mathbf{D} &= \mathbf{R}^e\dot{\boldsymbol{\varepsilon}}^e(\mathbf{R}^e)^T + \mathbf{R}^e\mathbf{D}^p(\mathbf{R}^e)^T, \\ \mathbf{W} &= \dot{\mathbf{R}}^e(\mathbf{R}^e)^T + \mathbf{R}^e\mathbf{W}^p(\mathbf{R}^e)^T, \end{aligned} \tag{5}$$

where $\mathbf{D}, \mathbf{R}^e\dot{\boldsymbol{\varepsilon}}^e(\mathbf{R}^e)^T$ and $\mathbf{D}^\mathbf{p}$ denote the symmetric parts, and $\mathbf{W}, \dot{\mathbf{R}}^e(\mathbf{R}^e)^T$ and $\mathbf{W}^\mathbf{p}$ stand for the skew-symmetric parts of the tensors $\mathbf{L}, \mathbf{L}^\mathbf{e}$ and $\mathbf{L}^\mathbf{p}$, respectively. For general problems related to finite elasto-plastic materials subject to large deformations, we refer the reader to the books by Mandel (1972), Lubliner (1990), Besseling and Van (1993) and Khan and Huang (1995). The papers by Dafalias (1985), Dafalias (1993), Dafalias and Rashid (1989), Van der Giessen (1991) and Kuroda (1995) focus on the concepts of plastic spin and its constitutive description for large strain elasto-plasticity. The notion of plastic spin is also considered in Badreddine et al. (2010) in the finite elasto-plastic model with a non-quadratic yield function and non-associated flow rule which is adapted to mixed hardening in the context of the ductile damage. The notion of plastic spin is also considered in Badreddine et al. (2010) in the finite elasto-plastic model with a non-quadratic yield function and non-associated flow rule which is adapted to mixed hardening in the context of the ductile damage.
Han et al. (2002) mention: "although the reorientation of anisotropic directions seems apparently for steel sheet metals, a proper computational treatments for practical applications particularly in sheet forming processes are quite rare." Han et al. (2002) consider the Lee type multiplicative decomposition of the deformation gradient, see Lee (1983) and the comments made by Cleja-Ţigoiu (1990),

$$\mathbf{F} = \mathbf{V}^e\mathbf{R}_*\mathbf{U}^p, \tag{6}$$

with $\mathbf{R}_* = \mathbf{R}^e\mathbf{R}^p$ derived as a composition of the previously introduced elastic and plastic rotations. Thus in the case of small elastic strains one obtains $\mathbf{F} \equiv \mathbf{R}\mathbf{U} = \mathbf{R}_*\mathbf{U}^p$, i.e., $\mathbf{R} = \mathbf{R}_*, \mathbf{U} = \mathbf{U}^p$. The directional axes of the yield rotate according to $\dot{\mathbf{R}}\mathbf{R}^{-1} = \mathbf{W} - \boldsymbol{\omega}^p$ with the expression for the plastic spin in the actual configuration taken from Kuroda, 1997. The model of associative plasticity is described in terms of Mandel's stress tensor, which is assumed to be symmetric. The symmetry of the Mandel stress tensor does not generally hold. For the discussion concerning the dissipative restrictions in finite anisotropic elasto-plasticity we make reference to Lubliner (1990), Cleja-Ţigoiu (2003).

- The presence of the plastic spin, $\mathbf{W}^p$, in these models makes possible the description of the orientational anisotropy, namely the change in time of the orthotropy direction, see Kim and Yin (1997) and Dafalias (2000). In the model proposed herein, the orthotropy directions are changing during the elasto-plastic process and they are characterized by Euler's angles. We adapt the representation of the plastic spins proposed by Cleja-Ţigoiu (2000a), Cleja-Ţigoiu (2007) also used in the paper by Cleja-Ţigoiu and Iancu (2011), to the model dependent on the third invariant of the stress.
- The associated flow rule describes the plastic stretching in terms of the constitutive function $\hat{\mathbf{N}}^p$, while the plastic spin characterizes the skew-symmetric part of the plastic distorsion-rate tensor in terms of $\hat{\boldsymbol{\Omega}}^p$ and with respect to the actual configuration. We make herein an attempt to develop a hardening model, which includes the kinematic hardening with Armstrong and Frederick (1966) law adapted to orthotropic materials, and isotropic hardening, and complies with the experimentally observed plastic yield and flow behaviour in cyclic loading reported by Verma et al. (2011), and orientational anisotropy emphasized by Kim and Yin (1997) and Hahm and Kim (2008). In Cleja-Ţigoiu (2007) and Cleja-Ţigoiu and Iancu (2011), a quadratic yield function and kinematic hardening were considered only.
- By pushing away to the actual configuration the material response, the constitutive and evolution functions become functions dependent on the Cauchy stress, tensorial hardening variable and orientational axes of orthotropy. The change in time of the orthotropic axes, $\mathbf{n}_i$, is characterized by the elastic rotation, $\mathbf{R}^e$, namely $\mathbf{m}_i = \mathbf{R}^e\mathbf{n}_i$, with the elastic spin, $\boldsymbol{\omega}^e = \dot{\mathbf{R}}^e(\mathbf{R}^e)^T$, expressed as a consequence of the kinematic relationships by $\boldsymbol{\omega}^e = \mathbf{W} - \mathbf{R}^e(\{\dot{\mathbf{P}}(\mathbf{P})^{-1}\}^a)(\mathbf{R}^e)^T$. The motion of the orthotropy axes is described in terms of Euler's angles, ψ, θ and φ, which characterize the elastic rotation. We avoid herein the discussion concerning the presence of a certain plastic spin related to the substructure and which is different from the kinematic plastic spin, see the point of view of Dafalias (1985), Ulz (2011) and Han et al. (2002).
- Once the complete set of rate type evolution equations has been defined in such a way to achieve the compatibility with the experimental data, the rate type boundary value problem can be solved using a variational inequality, together with the update procedure provided by Cleja-Ţigoiu and Matei (2012).
- Note that the representation theorems for anisotropic invariants provided by Liu (1982) and those corresponding to isotropic invariants given by Wang (1970) are applied everywhere in the present paper. We recall here a basic result.

Theorem 1.

1. *A function f is invariant with respect to the group g_6 in the isoclinic configuration, i.e., that f is an orthotropic function, if and only if there exists a function, say $\hat{f}$, which is isotropic with respect to the set of all variables mentioned below and given such that*

$$f(\mathbf{B}) = \hat{f}(\mathbf{B}, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2), \tag{7}$$

holds for any $\mathbf{B}$ *in the definition domain of function f and the tensorial orientational variables from the definition of the symmetry group.*

2. *For any elastic rotation* $\mathbf{R}^e \in Ort$

$$\mathbf{R}^e \hat{f}(\mathbf{B}, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2)(\mathbf{R}^e)^T = \hat{f}(\hat{\mathbf{B}}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2), \tag{8}$$

where either $\hat{\mathbf{B}} := \mathbf{R}^e \mathbf{B} (\mathbf{R}^e)^T$ *if* $\mathbf{B} \in Lin$ *or* $\widehat{B} = B$ if $B \in R$, *while* $\mathbf{m}_i = \mathbf{R}^e \mathbf{n}_i$. *Function* $\hat{f}$ *in the actual configuration is also isotropic, namely* $\forall \mathbf{Q} \in Ort$ *the following identity holds*

$$\mathbf{Q}\hat{f}(\hat{\mathbf{B}}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)(\mathbf{Q})^T = \hat{f}(\mathbf{Q}\hat{\mathbf{B}}(\mathbf{Q})^T, \mathbf{Q}\mathbf{m}_1 \otimes \mathbf{Q}\mathbf{m}_1, \mathbf{Q}\mathbf{m}_2 \otimes \mathbf{Q}\mathbf{m}_2). \tag{9}$$

As a final **remark**, if f is a scalar valued function, then $\hat{f}$ is given by isotropic invariants built with the set of tensorial fields mentioned above, see for instance the expressions of the yield function (25) and (26). If f is a tensor valued function, then $\hat{f}$ is represented in an appropriate tensorial basis with the coefficients expressed like scalar orthotropic functions, see for instance the constitutive expression for the plastic spin representation (35) and the constitutive function which describe the hardening (38).

Although we do not deal here with the problem related to the thermodynamics of irreversible processes, in the case of the isothermal processes, we make reference to the paper by Cleja-Ţigoiu (2003), concerning this subject within the constitutive framework of finite elasto-plasticity. The existence of the stress potential (i.e., the material with the hyperelastic property), as well as the reduced dissipation inequality, under the form of the so-called *principle of maximum plastic dissipation* were provided, based on the Il'yushin type dissipation postulate formulated by Cleja-Ţigoiu (2003). The consequences of the dissipation postulated were analyzed in relationships with Drucker's postulate (just in condition of isotropic material given by Lucchesi and Podio-Guidugli (1990)), Lubliner's flow rule in Lubliner (1990) and standard dissipative models, see Nguyen (1994). In this paper we suppose that the plastic stretching is directed to the normal to the yield surface, $\hat{\mathcal{F}}(\boldsymbol{\pi}, \boldsymbol{\alpha}) = 0$, $\boldsymbol{\alpha}$ being a stress-like tensorial internal variable, while the evolution of the tensorial hardening variable has been postulated in the form Armstrong and Frederick (1966) rule, adapted to the orthotropic materials, hence the full associated flow rule does not occur.

Notations

Lin and Lin_+ denote the set of all second order tensors and the invertible ones, respectively, while $Sym \subset Lin$ is the set of symmetric tensors.

$\mathbf{F} \in Lin_+$ is the deformation gradient.

$\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ are the initial orthotropy axes; $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ denote the actual orthotropy axes, namely $\mathbf{m}_i = \mathbf{R}^e \mathbf{n}_i$.

$\frac{D}{Dt}(\cdot)$ is the objective derivative associated with the elastic spin.

$\mathbf{A} \cdot \mathbf{B}$ is the scalar product between $\mathbf{A}, \mathbf{B} \in Lin$ represented by $\mathbf{A} \cdot \mathbf{B} = A_{ij}B_{ij}$ in terms of the in Cartesian components of the tensors, while $\mathbf{a} \cdot \mathbf{b} \equiv a_i b_i$ is the scalar product between the vectors $\mathbf{a}$, $\mathbf{b}$; for any $\mathbf{A} \in Lin$ the trace tr$\mathbf{A}$ is defined as the real number given by tr$\mathbf{A} = A_{ii}$, in terms of the Cartesian components A_{ij}.

$\mathbf{a} \otimes \mathbf{b} \in Lin$ denotes the tensorial product of the vectors $\mathbf{a}$ and $\mathbf{b}$, defined for any vector $\mathbf{v}$ by $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$.

$\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{A}$ is the kinematic hardening variable, and κ is the scalar hardening variable in the actual configuration.

$\boldsymbol{\pi}$ is the Piola–Kirchhoff symmetric tensor, $\boldsymbol{\alpha}$ is the tensorial hardening variable in the relaxed configuration.

$\mathbf{a} \otimes \mathbf{b}$ denotes the tensorial product of vectors $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{a} \otimes \mathbf{b} = a_i b_j \mathbf{i}_i \otimes \mathbf{i}_j$.

$\hat{\mathcal{F}}$ is the yield function; $\bar{f}$ is the error function.

$\hat{\mu}, \beta$ are the plastic factors; h_c is the hardening parameter.

σ_Y is the yield stress in tensile test.

$\hat{\mathcal{E}}$ is the fourth order elasticity tensor given for orthotropic case in Appendix (A4); a_{ij} are the elastic material constants; K_{ij} are yield parameters characterizing second order terms; B_k are yield parameters referring to third order terms; c_k, d_k, x_c, y_c, z_c are hardening constants; $A_j, \eta_k, \tilde{\eta}_k$ are plastic spin constants.

ψ, θ, φ are Euler's angles, namely the proper rotation φ, nutation θ, and precession ψ.

$\mathcal{H}$ is the Heaviside function, i.e., $\mathcal{H}(x) = 1$ if $x \geqslant 0$, and $\mathcal{H}(x) = 0$ if $x < 0$.

$<\beta> = \frac{1}{2}(\beta + |\beta|)$ denotes the positive part of the real function β.

Due to the large diversity existing in the terminology corresponding to finite elasto-plasticity, we adopted the most frequently used expressions, which also cover the content of the paper, as can be found, for instance, in the book by Gurtin et al. (2010):

$\mathbf{D} = \{\mathbf{L}\}^s$ and $\mathbf{D}^e = \{\mathbf{L}^e\}^s$ denote the symmetric parts, and are called the stretching and the elastic stretching, respectively, while $\mathbf{W} = \{\mathbf{L}\}^a$, and $\mathbf{W}^e = \{\mathbf{L}^e\}^a$ are the skew-symmetric part of the appropriate tensorial fields and they are called the spin and the elastic spin, respectively.

$\mathbf{R}^e$ is the elastic rotation; $\boldsymbol{\varepsilon}^e$ is the small elastic strain; $\boldsymbol{\omega}^e$ is the elastic spin; $\boldsymbol{\Omega}^p$ is the plastic spin.

$\mathbf{E}$ and $\mathbf{P}$ denote the elastic and plastic distortions.

$\mathbf{L}^e, \mathbf{L}^p$ are called elastic and plastic distortion-rate tensors.

$\mathbf{U}^e, \mathbf{V}^e$ symmetric and positive definite tensors which enter the polar decomposition of the elastic distortion, namely $\mathbf{E} = \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e$, are called elastic stretch tensors.

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2. Constitutive framework in the deformed configuration

In this paper, first we describe the behaviour of material with respect to local, relaxed, isoclinic configurations and the constitutive and evolution functions are assumed to be invariant with respect to the orthotropic symmetry group, g_6. We denote by $\boldsymbol{\pi}, \boldsymbol{\alpha}, \kappa, \boldsymbol{\varepsilon}^e$ and $\mathbf{n}_i$ the Piola–Kirchhoff stress tensor, tensorial internal variable, small elastic strain tensor and orthotropy direction in $\mathcal{K}$, respectively. The internal variables used in the model proposed herein, assumed to be a symmetric and traceless tensor of Piola–Kirchhoff type (usually called back stress), $\boldsymbol{\alpha}$, which is a stress-like variable and a scalar field κ, which can be either the equivalent plastic strain or the plastic work, see Khan and Huang (1995), Chung and Park (accepted for publication). It was clearly mentioned that α characterizes the translation motion of the yield surface in the stress space, while κ characterizes the change in the shape of the yield surface.

As a consequence of the representation Theorem 1 we introduce the following assumptions:

Ax. The material response is linear elastic with respect to the isoclinic configuration and is expressed in terms of the Piola–Kirchhoff symmetric stress tensor by

$$\boldsymbol{\pi} = \mathcal{E}(\boldsymbol{\varepsilon}^e) \equiv \widehat{\mathcal{E}}(\mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2)[\boldsymbol{\varepsilon}^e]. \tag{10}$$

The the linear orthotropic elastic type constitutive is written in the formula (A4) in the orthotropic basis.

- The plastic distortion-rate tensor is assumed to be given in terms of its symmetric and skew-symmetric parts, $\hat{\mathbf{N}}^p$ and $\hat{\boldsymbol{\Omega}}^p$, respectively, namely

$$\frac{d}{dt}\mathbf{P}(\mathbf{P})^{-1} = \hat{\mu}(\hat{\mathbf{N}}^p(\boldsymbol{\pi}, \boldsymbol{\alpha}, \kappa, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) + \hat{\boldsymbol{\Omega}}^p(\boldsymbol{\pi}, \boldsymbol{\alpha}, \kappa, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2)) \tag{11}$$

Remark. The plastic distortion-rate $\dot{\mathbf{P}}(\mathbf{P})^{-1}$ is power conjugated in isoclinic configuration with the Mandel stress measure, $\boldsymbol{\Sigma} = \mathbf{C}^e\boldsymbol{\pi}, \mathbf{C}^e = \mathbf{E}^T\mathbf{E}$, which is not generally a symmetric tensor. This result follows from the expression of the internal power $\mathbf{T}\cdot\mathbf{L}$, if the velocity gradient is replaced by its expression (4). If the elastic stretch tensor, $\mathbf{U}^e$, is small (just this is the case in this paper), then $\boldsymbol{\Sigma} \simeq \boldsymbol{\pi}$. Thus the Mandel's type tensor coincides with the Piola–Kirchhoff stress tensor in the isoclinic configuration. This is the rationale for using the symmetric Piola–Kirchhoff as a measure of the stress in the expression of the constitutive functions, expressed with respect to the isoclinic configuration.

- We add the evolution equations for the hardening variables

For any given smooth history of the deformation $t \to \mathbf{F}(t)$, the time evolution of the fields $\{\boldsymbol{\alpha}, \kappa\}$, for fixed $\mathbf{n}_k$, is described by the differential type system

$$\begin{aligned} &\dot{\boldsymbol{\alpha}} = \hat{\mu}\hat{\mathbf{l}}(\boldsymbol{\pi}, \boldsymbol{\alpha}, \kappa, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2),\\ &\dot{\kappa} = \hat{\mu}\hat{b}(\boldsymbol{\pi}, \boldsymbol{\alpha}, \kappa, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2),\\ &\dot{\mathbf{n}}_k = 0, \quad k = \{1,2,3\} \end{aligned} \tag{12}$$

Since the local, relaxed configurations are considered to be isoclinic, the orientation of these configurations is kept unchanged during the process, i.e., $\dot{\mathbf{n}}_k = 0$.

- Here $\hat{\mu}$ is the plastic multiplier associated with the yield criterion and is defined in terms of the yield function $\hat{\mathcal{F}}$

$$\hat{\mathcal{F}}(\boldsymbol{\pi}, \boldsymbol{\alpha}, \kappa, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) = 0 \tag{13}$$

such that the following relations are satisfied

$$\begin{aligned} &\hat{\mu} \geqslant 0, \hat{\mathcal{F}} \leqslant 0, \hat{\mu}\hat{\mathcal{F}} = 0 \text{ Khun—Tucker condition}\\ &\hat{\mu}\dot{\hat{\mathcal{F}}} = 0 \text{ consistency condition} \end{aligned} \tag{14}$$

An explicit formula can be given for the plastic multiplier, $\hat{\mu}$, written by using the evolution equation for the plastic distortion and the hardening variable in terms of the stretching $\mathbf{D}$ can be derived as a consequence of the consistency condition, namely

$$\begin{aligned} &\hat{\mu} = \frac{1}{h_c} < \beta > \mathcal{H}(\hat{\mathcal{F}}),\\ &\beta = \hat{\mathcal{E}}[\hat{\mathbf{N}}^p]\cdot\mathbf{D}, \quad h_c = \hat{\mathcal{E}}[\hat{\mathbf{N}}^p]\cdot\hat{\mathbf{N}}^p + \hat{\mathbf{N}}^p\cdot\hat{\mathbf{l}} - \partial_\kappa\hat{\mathcal{F}}\hat{b}, \end{aligned} \tag{15}$$

if the yield function is dependent on the effective stress $\boldsymbol{\pi} - \boldsymbol{\alpha}$ and the hardening parameter $\hat{h}_c$ is positive.

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The fields $\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\varepsilon}^e, \mathbf{n}_i$ pushed forward to the actual configuration are defined by

$$\mathbf{T} = \mathbf{R}^e \boldsymbol{\pi} (\mathbf{R}^e)^T, \quad \mathbf{A} = \mathbf{R}^e \boldsymbol{\alpha} (\mathbf{R}^e)^T, \quad \overline{\boldsymbol{\varepsilon}}^e := \mathbf{R}^e \boldsymbol{\varepsilon}^e (\mathbf{R}^e)^T, \quad \mathbf{m}_i = \mathbf{R}^e \mathbf{n}_i, \tag{16}$$

since the elastic rotation characterizes the passage from the isoclinic configuration to the actual configuration. These fields have the meaning of Cauchy stress, internal tensorial variable, small elastic strain and actual orientational variable, respectively, all of these written with respect to the actual configuration.

When we take the time derivative of (10) we get

$$\begin{aligned} &\frac{d}{dt}\boldsymbol{\pi} = \hat{\mathcal{E}}(\mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2)\left[\frac{d}{dt}(\boldsymbol{\varepsilon}^e)\right] \quad \Longleftrightarrow \\ &\mathbf{R}^e\left(\frac{d}{dt}\boldsymbol{\pi}\right)(\mathbf{R}^e)^T = \mathbf{R}^e \hat{\mathcal{E}}(\mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2)\left[\frac{d}{dt}(\boldsymbol{\varepsilon}^e)\right](\mathbf{R}^e)^T, \end{aligned} \tag{17}$$

using the linearity of the elastic constitutive equation. As a direct consequence of the Theorem 1 from the last rate type constitutive equation derived in (17) we derive in the actual configuration

$$\mathbf{R}^e\left(\frac{d}{dt}\boldsymbol{\pi}\right)(\mathbf{R}^e)^T = \hat{\mathcal{E}}(\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)\left[\mathbf{R}^e(\frac{d}{dt}(\boldsymbol{\varepsilon}^e))(\mathbf{R}^e)^T\right] \tag{18}$$

Further, when the fields are pushed away to the actual configuration, the objective time derivatives for the appropriate transformed fields $\mathbf{T}, \mathbf{A}, \overline{\boldsymbol{\varepsilon}}, \mathbf{m}_i$ appear through the following relationships

$$\begin{aligned} &\frac{D}{Dt}(\mathbf{T}) = \dot{\mathbf{T}} - \boldsymbol{\omega}^e\mathbf{T} + \mathbf{T}\boldsymbol{\omega}^e \equiv \mathbf{R}^e\dot{\boldsymbol{\pi}}(\mathbf{R}^e)^T \\ &\frac{D}{Dt}\mathbf{m}_k := \dot{\mathbf{m}}_k - \boldsymbol{\omega}^e\mathbf{m}_k, \quad \text{where} \quad \boldsymbol{\omega}^e = \dot{\mathbf{R}}^e(\mathbf{R}^e)^{-1} \end{aligned} \tag{19}$$

Relations similar to $\mathbf{T}$ also hold for $\boldsymbol{\varepsilon}$ and $\mathbf{A}$. The objective derivative of the fields $\mathbf{T}, \mathbf{A}, \overline{\boldsymbol{\varepsilon}}^e$, and $\mathbf{m}_i$ defined in the actual configuration are associated with the elastic spin.

The plastic distortion-rate tensor is pushed forward to the actual configuration, and its symmetric part, i.e., the plastic stretching, in the actual is given by

$$\mathbf{R}^e\mathbf{D}^p(\mathbf{R}^e)^T = \hat{\mu}\hat{\mathbf{N}}^p(\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2), \tag{20}$$

while its skew-symmetric part, i.e., the plastic spin, is postulated to be written as

$$\mathbf{R}^e\mathbf{W}^p(\mathbf{R}^e)^T = \hat{\mu}\hat{\boldsymbol{\Omega}}^p(\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2). \tag{21}$$

The elastic spin is defined as a consequence of the relationships (5) and (21)

$$\boldsymbol{\omega}^e = \mathbf{W} - \hat{\mu}\hat{\boldsymbol{\Omega}}^p(\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2). \tag{22}$$

We eliminate the rate of small elastic strain from the rate type constitutive equation derived in (18), using the kinematic relationship (5) together with (20). If we add the evolution equations for the hardening variables, again pushed away to the actual configuration, the following statements hold:

Theorem 2. *For any given smooth history of the deformation* $t \to \mathbf{F}(t)$, *the time evolution of the fields* $\{\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m_k}\}$ *is described by the differential type system*

$$\begin{aligned} &\frac{D}{Dt}\mathbf{T} = \hat{\mathcal{E}}(\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)[\mathbf{D}] - \hat{\mu}\hat{\mathcal{E}}(\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)[\hat{\mathbf{N}}^p], \\ &\frac{D}{Dt}\mathbf{A} = \hat{\mu}\hat{\mathbf{l}}, \quad \dot{\kappa} = \hat{\mu}\hat{b}, \quad \frac{D}{Dt}\mathbf{m}_k = 0, \end{aligned} \tag{23}$$

where the objective derivative $\frac{D}{Dt}$ *acts on the fields defined by formulae (19) together with (22). The arguments of the hat function are* $\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2$.

The initial conditions are introduced in the form $\mathbf{T}(t_0) = \mathbf{0}, \mathbf{A}(t_0) = \mathbf{0}, \kappa(t_0) = 0, \mathbf{m}_i(t_0) = \mathbf{n}_i, i = 1, 2, 3$.

Further, we do not mention the arguments of the hat function, namely $\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2$ or $(\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)$, unless this is necessary.

The effective stress $\overline{\mathbf{S}} = \mathbf{T} - \mathbf{A}$ is introduced since $\mathbf{A}$ has the meaning of the *back stress*. The back stress influences the position of the yield surface in the stress space.

$\overline{S}_{ij} = \mathbf{m}_i \cdot (\mathbf{T} - \mathbf{A})\mathbf{m}_j \;\; i, j = 1, 2, 3$, denote the components of the fields with respect to the actual orientational axes.

3. Yield function dependent on the third order invariant

We introduce a Drucker type yield function, which is dependent on the effective stress, i.e., $\overline{\mathbf{S}} = \mathbf{T} - \mathbf{A}$, and the scalar hardening κ, in a new approach based on the g_6- invariance assumption.

Ax. Yield function is represented in terms of three scalar valued functions as

$$\hat{\mathcal{F}} := (\hat{f}_2)^{3/2} - \gamma \hat{f}_3 - F = 0, \tag{24}$$

where

- $\hat{f}_2$ is g_6 invariant and homogeneous of the second degree, while
- $\hat{f}_3$ is g_6 invariant and homogeneous of the third degree with respect to the effective stress;
- function $F = F(\kappa)$ describes the scalar hardening and depends on κ only.

The yield function written on the left hand side of (24) has an expression suggested by the initial like yield condition proposed by Cazacu and Barlat (2001) and used in Cazacu and Barlat (2004) and Nixon et al. (2010).

Here we use the representation theorems of Liu (1982) and Wang (1970) and, consequently, the g_6 invariant functions f_2 and f_3 will be described as functions dependent on the scalar invariants generated by $\overline{\mathbf{S}} \in Sym$ and orientational variables $\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2 \in Sym$, as we mentioned in Theorem 1.

Theorem 3. *The homogeneous of the second degree and g_6 invariant function f_2 is represented using nine material parameters C_i*

$$\begin{aligned}\hat{f}_2(\mathbf{T}, \mathbf{A}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2) = {} & C_1 \overline{\mathbf{S}} \cdot \overline{\mathbf{S}} + C_2 \overline{\mathbf{S}}^2 \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1) + C_3 \overline{\mathbf{S}}^2 \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2) + C_4 (\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1))^2 + C_5 (\overline{\mathbf{S}} \cdot (\mathbf{m}_2 \\ & \otimes \mathbf{m}_2))^2 + C_6 (\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1))(\overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)) + C_7 (tr(\overline{\mathbf{S}}))^2 + C_8 tr(\overline{\mathbf{S}})(\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1)) \\ & + C_9 tr(\overline{\mathbf{S}})(\overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2))\end{aligned} \tag{25}$$

2. The homogeneous of third degree and g_6 invariant function f_3 allows a representation in terms of the basic set of invariants involving nineteen material constants

$$\begin{aligned}\hat{f}_3(\mathbf{T}, \mathbf{A}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2) = {} & B_1 (\overline{\mathbf{S}} \cdot \overline{\mathbf{S}})(\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1)) + B_2 (\overline{\mathbf{S}} \cdot \overline{\mathbf{S}})(\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1)) + [\overline{\mathbf{S}}^2 \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1)](B_3 \overline{\mathbf{S}} \cdot (\mathbf{m}_1 \\ & \otimes \mathbf{m}_1) + B_4 \overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)) + [\overline{\mathbf{S}}^2 \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)](B_5 \overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1) + B_6 \overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)) \\ & + B_7 (\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1))^2 (\overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)) + B_8 (tr(\overline{\mathbf{S}}))^3 + B_9 tr(\overline{\mathbf{S}}^3 (\mathbf{m}_1 \otimes \mathbf{m}_1)) \\ & + B_{10} tr(\overline{\mathbf{S}}^3 (\mathbf{m}_2 \otimes \mathbf{m}_2)) + B_{11} (\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1))(\overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2))^2 + B_{12} (\overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1))^3 \\ & + B_{13} (\overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2))^3 + tr(\overline{\mathbf{S}})(B_{14} \overline{\mathbf{S}}^2 \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1) + B_{15} \overline{\mathbf{S}}^2 \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)) \\ & + (tr\overline{\mathbf{S}})^2 (B_{16} \overline{\mathbf{S}} \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1) + B_{17} \overline{\mathbf{S}} \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)) + B_{18} tr(\overline{\mathbf{S}}^3) + B_{19} tr(\overline{\mathbf{S}}) tr(\overline{\mathbf{S}}^2)\end{aligned} \tag{26}$$

Generally, the material parameters $C_i, i \in \{1, \ldots, 9\}$ and $B_i\ i \in \{1, \ldots 19\}$ could depend on the scalar hardening variable κ. Here we restrict ourselves at the hypothesis that C_i, B_i are constant.

Remarks:

Some peculiar properties of the yield function (24) could be emphasized:

- Apart from the yield function considered in Cleja-Ţigoiu (2007), f_2 contains all g_6 invariants that generate a second order homogeneous function with respect to the effective stress. If we compare f_2 with the yield function introduced by Cleja-Ţigoiu (2007), which is independent of $tr(\overline{\mathbf{S}})$, we can conclude that the two yield functions coincide if and only if $C_7 = C_8 = C_9 = 0$ and $\gamma = 0$. Hill's quadratical yield function with mixed hardening but no evolution of the orthotropic axes is used in Vladimirov et al. (2011) and Tang et al. (2008).
- For $\gamma \neq 0$, the third invariant of the tensor $\overline{\mathbf{S}}$ is involved in the definition of the yield function.
- Cazacu and Barlat (2004) proposed the yield function in the form $(J_2)^{3/2} - c\, J_3 = \tau_Y^3$. In the aforementioned paper, f_2 and f_3 are expressed by the stress components with respect to the orthotropy directions (in the deformed configuration), and not in terms of the appropriate invariants of the stress. There is no evolution equation which could describe the motion of the orthotropy axes with respect to time. Consequently, the yield function is viewed in the fixed axes since the motion of the orthotropy axes with respect to time is not described at all.

Function f_2 allows a representation with respect to components $\overline{S}_{ij}$ of the following type

$$\begin{aligned}\hat{f}_2(\mathbf{T}, \mathbf{A}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2) = {} & K_{11} \overline{S}_{11}^2 + K_{22} \overline{S}_{22}^2 + K_{33} \overline{S}_{33}^2 + K_{m1} \overline{S}_{12}^2 + K_{m2} \overline{S}_{13}^2 + K_{m3} \overline{S}_{23}^2 + K_{12} \overline{S}_{11} \overline{S}_{22} + K_{13} \overline{S}_{11} \overline{S}_{33} \\ & + K_{23} \overline{S}_{22} \overline{S}_{33}\end{aligned} \tag{27}$$

where the new material parameters K_{ij}, as functions of $\{C_i, i \ldots 9\}$, are presented in (A1).

Proposition 1. *The sufficient conditions that ensure the yield function to be pressure insensitive can be derived from the restrictions*

$$\hat{f}_j(\bar{\mathbf{S}} - p\mathbf{I}, \mathbf{m_1} \otimes \mathbf{m_1}, \mathbf{m_2} \otimes \mathbf{m_2}) = \hat{f}_j(\bar{\mathbf{S}}, \mathbf{m_1} \otimes \mathbf{m_1}, \mathbf{m_2} \otimes \mathbf{m_2}), \quad \text{for } j = 2, 3, \tag{28}$$

$\forall\, \bar{\mathbf{S}} \in Sym$, *as the relationships between yield constants*

$$K_{12} = K_{33} - K_{11} - K_{22}, \quad K_{13} = K_{22} - K_{11} - K_{33}, \quad K_{23} = K_{11} - K_{22} - K_{33} \tag{29}$$

and

$$\begin{aligned} &B_{11} = -B_4 - B_5 - B_7,\ B_{12} = (1/3)(-2B_3 - B_7), B_{13} = (1/3)(B_4 + B_5 - 2B_6 + B_7),\\ &B_{14} = (1/3)(-B_3 - B_4 - 3B_9), \quad B_{15} = (1/3)(-B_5 - B_6 - 3B_{10}),\\ &B_{16} = (1/9)(-3B_1 + B_3 + B_4 + 3B_9),\\ &B_{17} = (1/9)(-3B_2 + B_5 + B_6 + 3B_{10}),\\ &B_{18} = (1/18)(-9B_1 - 9B_2 + B_3 + B_4 + B_5 + B_6 + 81B_8 + 3B_9 + 3B_{10}),\\ &B_{19} = (1/18)(3B_1 + 3B_2 - B_3 - B_4 - B_5 - B_6 - 81B_8 - 3B_9 - 3B_{10}), \end{aligned} \tag{30}$$

hold.

Remark The initial yield function (24) for a pressure insensitive material is characterized by sixteen yield constants, out of which six refer to function f_2, namely K_{ij} introduced by (27) together with (29), and ten refer function f_3, namely $B_k, k = 1, \ldots 10$.

Proposition 2.

1. *Function $\hat{f}_2$ has the following component representation*

$$\hat{f}_2 = \frac{1}{2}(K_{11} + K_{22} - K_{33})(\bar{S}_{11} - \bar{S}_{22})^2 + \frac{1}{2}(K_{11} + K_{33} - K_{22})(\bar{S}_{11} - \bar{S}_{33})^2 + \frac{1}{2}(K_{22} + K_{33} - K_{11})(\bar{S}_{22} - \bar{S}_{33})^2 + K_{m1}\bar{S}^2_{12} + K_{m2}\bar{S}^2_{13} + K_{m3}\bar{S}^2_{23} \tag{31}$$

2. *The pressure insensitive yield function (27), i.e., as given by (31), is positive definite if and only if*

$$K_{11} + K_{22} - K_{33} > 0, \quad K_{11} + K_{33} - K_{22} > 0, \quad K_{22} + K_{33} - K_{11} > 0, \quad K_{m1} > 0, \quad K_{m2} > 0, \quad K_{m3} > 0 \tag{32}$$

We can recover the quadratic expression for the components of the stress tensor with respect to the orthotropic axes, which enter the expression of the Hill (1948) criterion, if we introduce the notations in (31),

$$\begin{aligned} &F = \frac{1}{2}(K_{22} + K_{33} - K_{11}),\ G = \frac{1}{2}(K_{11} + K_{33} - K_{22}),\\ &E = \frac{1}{2}(K_{11} + K_{22} - K_{33}),\ 2L = K_{m3},\ 2M = K_{m2},\ 2N = K_{m1}. \end{aligned}$$

Proposition 3. *The pressure insensitive function $\hat{f}_3$ can be expressed in terms of the stress components $\bar{S}_{ij} = \mathbf{m}_i \cdot \bar{\mathbf{S}}\mathbf{m_j}$ as*

$$\begin{aligned} \hat{f}_3 = {} & k_1\bar{S}^3_{11} + k_2\bar{S}^3_{22} + k_3\bar{S}^3_{33} + k_4\bar{S}^2_{11}\bar{S}_{22} + k_5\bar{S}^2_{11}\bar{S}_{33} + k_6\bar{S}^2_{22}\bar{S}_{11} + k_7\bar{S}^2_{22}\bar{S}_{33} + k_8\bar{S}^2_{33}\bar{S}_{11} + k_9\bar{S}^2_{33}\bar{S}_{22} + k_{10}\bar{S}^2_{12}\bar{S}_{11}\\ & + k_{11}\bar{S}^2_{12}\bar{S}_{22} + k_{12}\bar{S}^2_{12}\bar{S}_{33} + k_{13}\bar{S}^2_{13}\bar{S}_{11} + k_{14}\bar{S}^2_{13}\bar{S}_{22} + k_{15}\bar{S}^2_{13}\bar{S}_{33} + k_{16}\bar{S}^2_{23}\bar{S}_{11} + k_{17}\bar{S}^2_{23}\bar{S}_{22} + k_{18}\bar{S}^2_{23}\bar{S}_{33} + k_{19}\bar{S}_{11}\bar{S}_{22}\bar{S}_{33}\\ & + k_{20}\bar{S}_{12}\bar{S}_{13}\bar{S}_{23} \end{aligned} \tag{33}$$

where the constants k_i, $i \in \{1, \ldots 20\}$ are given in (A2), (A3) and depend on the material constants previously introduced B_j, $j \in \{1, \ldots, 10\}$.

4. Orthotropic model dependent on the third invariant of the stress

1. The plastic distortion-rate tensor pushed forward to the actual configuration is proposed to be given by its symmetric and skew-symmetric parts, $\hat{\mathbf{N}}^p$ and $\hat{\mathbf{\Omega}}^p$, respectively, in (11).
2. The flow rule associated with the yield surface (24) defines the symmetric part of the plastic distortion-rate tensor, namely the plastic stretching, in the deformed configuration as

$$\mathbf{R}^e\mathbf{D}^p(\mathbf{R}^e)^T = \hat{\mu}\hat{\mathbf{N}}^p \text{ with } \quad \hat{\mathbf{N}}^p = \partial_{\mathbf{T}}\hat{\mathcal{F}} \equiv \frac{3}{2}\sqrt{\hat{f}_2}\partial_{\mathbf{T}}\hat{f}_2 - \gamma\partial_{\mathbf{T}}\hat{f}_3, \tag{34}$$

namely $(\partial_{\mathbf{T}}\hat{f}_k)_{ii} := \mathbf{m}_i \cdot (\partial_{\mathbf{T}}\hat{f}_k)\mathbf{m}_i = \partial_{T_{ii}}\hat{f}_k,\ i \in \{1, 2, 3\}$, and
$(\partial_{\mathbf{T}}\hat{f}_k)_{ij} := \mathbf{m}_i \cdot (\partial_{\mathbf{T}}\hat{f}_k)\mathbf{m}_j = \frac{1}{2}\partial_{T_{ij}}\hat{f}_k,\ i,j = \{1, 2, 3\},\ i \neq j.$

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3. Three types of plastic spins, namely three constitutive functions $\hat{\boldsymbol{\Omega}}^p$, are proposed herein. In order to provide the isotropic functions which take the skew-symmetric values, we use the representation theorems of Wang (1970).
   - On assuming that the plastic spin is generated by $\bar{\mathbf{S}}$ and the orientational variables $\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2$, which generalizes the Mandel type plastic spin (*called plastic spin I*), we introduce

$$\begin{aligned}\hat{\boldsymbol{\Omega}}^p = {} & A_1(\bar{\mathbf{S}}(\mathbf{m_1}\otimes\mathbf{m_1}) - (\mathbf{m_1}\otimes\mathbf{m_1})\bar{\mathbf{S}}) + A_2(\bar{\mathbf{S}}(\mathbf{m_2}\otimes\mathbf{m_2}) - (\mathbf{m_2}\otimes\mathbf{m_2})\bar{\mathbf{S}}) + A_3((\mathbf{m_2}\otimes\mathbf{m_2})\bar{\mathbf{S}}(\mathbf{m_1}\otimes\mathbf{m_1}) - (\mathbf{m_1} \\ & \otimes\mathbf{m_1})\bar{\mathbf{S}}(\mathbf{m_2}\otimes\mathbf{m_2})) + A_4(\bar{\mathbf{S}}^2(\mathbf{m_1}\otimes\mathbf{m_1}) - (\mathbf{m_1}\otimes\mathbf{m_1})\bar{\mathbf{S}}^2) + A_5(\bar{\mathbf{S}}^2(\mathbf{m_2}\otimes\mathbf{m_2}) - (\mathbf{m_2}\otimes\mathbf{m_2})\bar{\mathbf{S}}^2) + A_6((\mathbf{m_2} \\ & \otimes\mathbf{m_2})\bar{\mathbf{S}}^2(\mathbf{m_1}\otimes\mathbf{m_1}) - (\mathbf{m_1}\otimes\mathbf{m_1})\bar{\mathbf{S}}^2(\mathbf{m_2}\otimes\mathbf{m_2})).\end{aligned} \tag{35}$$

For $A_4 = A_5 = A_6 = 0$ the plastic spin is reduced to the linear expression with respect to the effective stress $\bar{\mathbf{S}}$. Such type of the plastic spin is firstly introduced by Dafalias (1985) and Loret (1983). In the class of orthotropic $\boldsymbol{\Sigma}-$ model developed by Cleja-Ţigoiu (2000a), a similar (with aforementioned papers) linear representation of the plastic has been proposed in terms of the non-symmetric Mandel stress measure $\boldsymbol{\Sigma}$. If we take $A_3 = A_4 = A_5 = A_6 = 0$, from the formula (35) we obtain the Mandel type plastic spin proposed by Cleja-Ţigoiu (2007).

   - If we assume that the plastic spin (called *plastic spin II*) is generated by $\hat{\mathbf{N}}^p$ and the orientational variables $\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2$, we obtain

$$\begin{aligned}\hat{\boldsymbol{\Omega}}^p = {} & \eta_1(\hat{\mathbf{N}}^p(\mathbf{m}_1\otimes\mathbf{m}_1) - (\mathbf{m}_1\otimes\mathbf{m}_1)\hat{\mathbf{N}}^p) + \eta_2(\hat{\mathbf{N}}^p(\mathbf{m}_2\otimes\mathbf{m}_2) - (\mathbf{m}_2\otimes\mathbf{m}_2)\hat{\mathbf{N}}^p) + \eta_3((\mathbf{m}_2\otimes\mathbf{m}_2)\hat{\mathbf{N}}^p(\mathbf{m}_1\otimes\mathbf{m}_1) \\ & - (\mathbf{m}_1\otimes\mathbf{m}_1)\hat{\mathbf{N}}^p(\mathbf{m}_2\otimes\mathbf{m}_2)) + \eta_4((\hat{\mathbf{N}}^p)^2(\mathbf{m}_1\otimes\mathbf{m}_1) - (\mathbf{m}_1\otimes\mathbf{m}_1)(\hat{\mathbf{N}}^p)^2) + \eta_5((\hat{\mathbf{N}}^p)^2(\mathbf{m}_2\otimes\mathbf{m}_2) - (\mathbf{m}_2\otimes\mathbf{m}_2) \\ & \times(\hat{\mathbf{N}}^p)^2) + \eta_6((\mathbf{m}_2\otimes\mathbf{m}_2)(\hat{\mathbf{N}}^p)^2(\mathbf{m}_1\otimes\mathbf{m}_1) - (\mathbf{m}_1\otimes\mathbf{m}_1)(\hat{\mathbf{N}}^p)^2(\mathbf{m}_2\otimes\mathbf{m}_2))\end{aligned} \tag{36}$$

Remark. When an associated flow rule with a quadratic yield function is considered, the representations (35) and (36) become equivalent. Formula (36) for $\eta_4 = \eta_5 = \eta_6 = 0$ is reduced to the Liu–Wang type spin proposed in Cleja-Ţigoiu (2007).

   - When the plastic spin is generated by $\bar{\mathbf{S}}$, $\hat{\mathbf{N}}^p$ and the orientational variables, the expression of the plastic spin (called *plastic spin III*) can be derived in the following form

$$\begin{aligned}\hat{\boldsymbol{\Omega}}^p = {} & \tilde{\eta}(\bar{\mathbf{S}}\hat{\mathbf{N}}^p - \hat{\mathbf{N}}^p\bar{\mathbf{S}}) + \tilde{\eta}_1(\bar{\mathbf{S}}\hat{\mathbf{N}}^p(\mathbf{m}_1\otimes\mathbf{m}_1) - (\mathbf{m}_1\otimes\mathbf{m}_1)\hat{\mathbf{N}}^p\bar{\mathbf{S}}) + \tilde{\eta}_2(\bar{\mathbf{S}}\hat{\mathbf{N}}^p(\mathbf{m}_2\otimes\mathbf{m}_2) - (\mathbf{m}_2\otimes\mathbf{m}_2)\hat{\mathbf{N}}^p\bar{\mathbf{S}}) + \tilde{\eta}_3((\mathbf{m}_2 \\ & \otimes\mathbf{m}_2)\hat{\mathbf{N}}^p\bar{\mathbf{S}}(\mathbf{m}_1\otimes\mathbf{m}_1) - (\mathbf{m}_1\otimes\mathbf{m}_1)\bar{\mathbf{S}}\hat{\mathbf{N}}^p(\mathbf{m}_2\otimes\mathbf{m}_2))\end{aligned} \tag{37}$$

For $\tilde{\eta}_k = 0, k = 1, 2, 3$, the *plastic spin III* is reduced to the non-coaxiality law, between $\bar{\mathbf{S}}$ and the plastic flow rule, that has been derived by Bammann and Aifantis (1987), Paulum and Percherski (1987), Zbib and Aifantis (1988), Van der Giessen (1991) and Kuroda (1995). Formula (37) does not contain an appropriate complete set of skew-symmetric invariants. If $\tilde{\eta}_1 = \tilde{\eta}_2 = \tilde{\eta}_3 = 0$, then the Dafalias type spin introduced in Cleja-Ţigoiu (2007), see also Cleja-Ţigoiu and Iancu (2011), can be derived. For comparison, we also refer the reader to Dafalias (2000).

4. We introduce an evolution equation for the tensorial hardening variable in the actual configuration as *an Armstrong–Frederick type hardening law* adapted to the orthotropic material

$$\begin{aligned}& \frac{D}{Dt}\mathbf{A} = \mu\hat{\mathbf{l}}(\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1\otimes\mathbf{m}_1, \mathbf{m}_2\otimes\mathbf{m}_2), \quad \text{where} \\ & \hat{\mathbf{l}} = c_0\hat{\mathbf{N}}^p + c_1[\hat{\mathbf{N}}^p(\mathbf{m}_1\otimes\mathbf{m}_1) + (\mathbf{m}_1\otimes\mathbf{m}_1)\hat{\mathbf{N}}^p] + c_2[\hat{\mathbf{N}}^p(\mathbf{m}_2\otimes\mathbf{m}_2) + (\mathbf{m}_2\otimes\mathbf{m}_2)\hat{\mathbf{N}}^p] - \hat{b}(\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1\otimes\mathbf{m}_1, \mathbf{m}_2 \\ & \quad \otimes\mathbf{m}_2)[d_0\mathbf{A} + d_1(\mathbf{A}(\mathbf{m}_1\otimes\mathbf{m}_1) + (\mathbf{m}_1\otimes\mathbf{m}_1)\mathbf{A}) + d_2(\mathbf{A}(\mathbf{m}_2\otimes\mathbf{m}_2) + (\mathbf{m}_2\otimes\mathbf{m}_2)\mathbf{A})]\end{aligned} \tag{38}$$

where $c_0, c_1, c_2, d_0, d_1, d_2$ are material constants.

Remark. Eq. (38) would become of the Prager type, (see Khan and Huang (1995), Chung and Park (accepted for publication)) and not of the Prager–Ziegler type, provided that the following equalities hold: $c_1 = c_2 = 0, d_0 = d_1 = d_2 = 0$. Eq. (38) can be reduced to the Prager type hardening law adapted to the orthotropic material in Cleja-Ţigoiu (2007) if $d_0 = d_1 = d_2 = 0$, which have been considered in Cleja-Ţigoiu and Iancu (2011). Moreover, if we consider that only c_0 and d_0 are non-vanishing parameters in Eq. (38), the non-associative hardening law $\frac{D\mathbf{A}}{Dt} = c_0\mu\,\partial_{\mathbf{T}}\mathcal{F} - d_0\mathbf{A}\dot{\kappa}$ can be derived. Within the constitutive framework of small strains, with a similar hardening law and parameters c_0 and d_0 as functions of κ, Chung and Park (accepted for publication) introduced and analyzed the corresponding consistency condition (which should not be considered as the consistency condition from rate-independent plasticity) of coupled isotropic and kinematic hardening with isotropic hardening under the proportional loading. However, our aim is to analyze a mathematical model that does not satisfy the aforementioned relations. Only if these five constants are not vanishing, i.e., when we take into account the influence of the orthotropy on the hardening, it is possible to determine their values to be compatible with the experimental data given by Verma et al. (2011).

5. The evolution equation for the scalar hardening variable, κ, in the actual configuration is given by

$$\dot{\kappa} = \hat{\mu}\hat{b}(\mathbf{T}, \mathbf{A}, \kappa, \mathbf{m}_1\otimes\mathbf{m}_1, \mathbf{m}_2\otimes\mathbf{m}_2). \tag{39}$$

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6. Two types of scalar hardening variables can be introduced in the model:
   (a) κ is the equivalent plastic strain and it can be given by the evolution equation

$$\dot{\kappa} = \sqrt{\hat{\mathbf{D}}^p(t) \cdot \hat{\mathbf{D}}^p(t)}, \tag{40}$$

with the appropriate scalar hardening function, i.e., Swift-type function, given by

$$F(\kappa) = \sigma_Y^3 (k\kappa + 1)^{3n}, \quad \text{when } \hat{b} = \sqrt{\hat{\mathbf{N}}^p \cdot \hat{\mathbf{N}}^p}. \tag{41}$$

   (b) κ evaluates the plastic work and it is characterized by

$$\dot{\kappa} = \bar{\mathbf{S}}(t) \cdot \hat{\mathbf{D}}^p(t), \tag{42}$$

with the associated scalar hardening functions prescribed by the Voce-type function

$$F(\kappa) = \sigma_Y^3 (x_c + y_c e^{-z_c\kappa})^3, \text{ when } \hat{b} = \frac{1}{\sigma_Y(x_c + y_c e^{-z_c\kappa})} \bar{\mathbf{S}} \cdot \hat{\mathbf{N}}^{\mathbf{p}} \tag{43}$$

where k, n, x_c, y_c, z_c are material constants.

Notice the essential difference, $\dot{\kappa}$ can not be negative during an elasto-plastic processes in the case (a), in contrast with the case (b), when the change in the sign of $\dot{\kappa}$ is allowed. Just the changing in the sign of $\dot{\kappa}$ during the processes in the case (b), could be useful to make the difference between tension and compression processes and could certificate the rationale in selection of the scalar hardening variable done by Verma et al. (2011). "In the simulation of metals forming, modelling the flow stress as an average behaviour of material over a deformation range is more important than determining an initial yield locus of the material," see the assertion in Verma et al. (2011).

Remarks. The scalar function $F(\kappa)$ of Swift-type has been used in Banabic et al. (2003), while Verma et al. (2011) used the scalar function of Voce-type, which are here adapted for homogeneous functions of the third degree with respect to the effective stress.

5. Orientational orthotropy and Euler angles

Herein, we resume the procedure previously introduced by Cleja-Ţigoiu and Iancu (2011) and which allows us to characterize the motion of the orthotropic axes during the deformation process in terms of three angles, namely the so-called Euler angles, see Beju et al. (1983).

Three set of othogonal axes is introduced

- $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$ represent the fixed orthonormal basis, say the geometric axes of a sheet,
- $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the initial orthotropy directions that characterizes the orthotropy direction in the so-called relaxed (or plastically deformed) configuration, generally different from the axes of the sheet,
- $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are the orthotropy directions in the actual configuration, which satisfy the initial condition $\mathbf{m}_i(t_0) = \mathbf{n}_i$, $i = 1, 2, 3$, see Fig. 1.

We denote by $\mathbf{R} \in Ort$ the rotation tensor which characterizes the position of the orthotropy axes $\mathbf{m}_i$ with respect to the fixed axes $\mathbf{j}_i$, namely $\mathbf{R}\mathbf{j}_k = \mathbf{m}_k, \quad k = 1, 2, 3$.

The initial position of the orthotropy axes $\mathbf{n}_i$ with respect to the fixed axes $\mathbf{j}_i$ is characterized by the rotation tensor, $\mathbf{R}_0 \in \; Ort$, namely $\quad \mathbf{R}_0(\mathbf{j}_k) = \mathbf{n}_k, \quad k = 1, 2, 3$.

The elastic rotation tensor is denoted by $\mathbf{R}^e \in \; Ort$ and it is is related to $\mathbf{R}$ by

$$\mathbf{R}^e\mathbf{n}_k = \mathbf{m}_k, k = 1, 2, 3, \quad \mathbf{R}^e(t) = \mathbf{R}(t)(\mathbf{R}_0)^{-1}, \quad \mathbf{R}^e(t_0) = \mathbf{I} \tag{44}$$

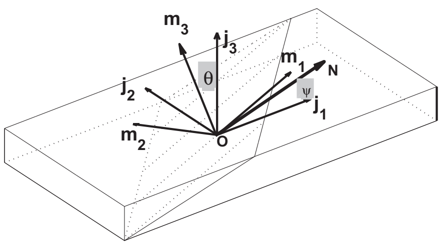

Fig. 1. The sheet and the orthotropy axes in actual configuration.

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The meaning of the Euler angles, which characterize the components $R_{ik} = \mathbf{j}_i \cdot \mathbf{R}\mathbf{j}_k$ of the rotation tensor $\mathbf{R}$, is described as follows:

- the nutation θ is the angle between the axes $\mathbf{m}_3$ and $\mathbf{j}_3$,
- the precession ψ is measured from $\mathbf{j}_1$ to the nodal axis denoted by ON, which is the intersection of the planes $(\mathbf{j}_1, \mathbf{j}_2)$ and $(\mathbf{m}_1, \mathbf{m}_2)$,
- the proper rotation φ is the angle between ON and the axis $\mathbf{m}_1$, see Fig. 1.

Let us remark that in the case of a plane rotation, when $\theta = 0$, the precession angle ψ is misleading and is considered to be zero.

The components of the rotation tensor $\mathbf{R}$ can be expressed in terms of the Euler angles by the following formulae

$$(R_{ik}) = \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ \cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & -\sin\theta\cos\varphi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix} \tag{45}$$

Note that the elastic spin $\omega^e = \dot{\mathbf{R}}^e\mathbf{R}^{\mathbf{e}T} = \dot{\mathbf{R}}\mathbf{R}^T$ with respect to the current orthotropic basis $\mathbf{m}_i \otimes \mathbf{m}_j$ has the components

$$(\omega^e_{ik}) = \begin{pmatrix} 0 & -\dot\varphi - \dot\psi\cos\theta & \dot\theta\sin\varphi - \dot\psi\sin\theta\cos\varphi \\ \dot\varphi + \dot\psi\cos\theta & 0 & -\dot\theta\cos\varphi - \dot\psi\sin\theta\sin\varphi \\ -\dot\theta\sin\varphi + \dot\psi\sin\theta\cos\varphi & \dot\theta\cos\varphi + \dot\psi\sin\theta\sin\varphi & 0 \end{pmatrix} \tag{46}$$

$\mathbf{R}_0$ is characterized by the initial values of the appropriate Euler angles $\psi(t_0), \varphi(t_0), \theta(t_0)$.

We introduce the notations for the components of the stretching tensor, $\mathbf{D}$, with respect to the axes $\{\mathbf{j}_k\}$ and $\{\mathbf{m}_k\}$, as well as the motion spin and tensorial hardening variable $\mathbf{A}$

$$D_{ik} = \mathbf{j}_i \cdot \mathbf{D}\mathbf{j}_k, \quad \widetilde{D}_{ij} = \mathbf{m}_i \cdot \mathbf{D}\mathbf{m}_j, \quad \widetilde{W}_{ij} = \mathbf{m}_i \cdot \mathbf{W}\mathbf{m}_j, \quad \hat{\Omega}^p_{ij} = \mathbf{m}_i \cdot \hat{\Omega}^p\mathbf{m}_j, \quad A_{ij} = \mathbf{m}_i \cdot \mathbf{A}\mathbf{m}_j. \tag{47}$$

Remark. As it has been noticed by Cleja-Ţigoiu (2007), the time derivative of the components of the Cauchy stress with respect to the actual orthotropic axes are just the projections of the objective derivative taken with respect to the elastic spin, namely

$$\frac{d}{dt}(\mathbf{m}_k \cdot \mathbf{T}\mathbf{m}_j) = \mathbf{m}_k \cdot \frac{D}{Dt}(\mathbf{T})\mathbf{m}_j. \tag{48}$$

As a direct consequence of this remark and using formula (22), the following result can be proven.

Notice the formal similarity of the differential system which allows to determine the state of the material, namely the current values of the Cauchy stress, hardening variables and Euler angles, for a given history of the deformation gradient with the appropriate one proved by Cleja-Ţigoiu and Iancu (2011). However the constitutive frameworks are completely different.

Theorem 4. *For a given smooth history of the deformation $t \to \mathbf{F}(t)$ at a fixed material point, the evolutions with respect to time of the components of the Cauchy stress, $\mathbf{T}$, and the tensorial hardening variable, $\mathbf{A}$, in the basis $\mathbf{m}_i \otimes \mathbf{m}_j$, the Euler angles, ψ, θ and φ, and the scalar hardening variable, κ, are described by the following differential system*

$$\begin{aligned} &\frac{d}{dt}T_{ij} = \mathbf{m}_i \cdot \hat{\mathcal{E}}[\mathbf{D}]\mathbf{m}_j - \hat{\mu}\mathbf{m}_i \cdot (\hat{\mathcal{E}})[\hat{\mathbf{N}}^p]\mathbf{m}_j \\ &\frac{d}{dt}A_{ij} = \hat{\mu}\mathbf{m}_i \cdot \hat{\mathbf{l}}\mathbf{m}_j \\ &\dot\varphi + \dot\psi\cos\theta = \mathbf{m}_2 \cdot (\mathbf{W} - \hat{\mu}\hat{\mathbf{\Omega}}^p)\mathbf{m}_1 \\ &-\dot\theta\sin\varphi + \dot\psi\sin\theta\cos\varphi = \mathbf{m}_3 \cdot (\mathbf{W} - \hat{\mu}\hat{\mathbf{\Omega}}^p)\mathbf{m}_1 \\ &\dot\theta\cos\varphi + \dot\psi\sin\theta\sin\varphi = \mathbf{m}_3 \cdot (\mathbf{W} - \hat{\mu}\hat{\mathbf{\Omega}}^p)\mathbf{m}_2 \\ &\dot\kappa = \hat{\mu}\hat{b} \end{aligned} \tag{49}$$

where $\mathbf{D} = \{\dot{\mathbf{F}}(\mathbf{F})^{-1}\}^s$ and $\mathbf{W} = \{\dot{\mathbf{F}}(\mathbf{F})^{-1}\}^a$. The above differential system is associated with the yield condition $\hat{\mathcal{F}} = 0$, while the plastic factor, $\hat{\mu}$, is defined by the formulae (15), while the initial conditions are given by

$$A_{ij}(t_0) = 0, \kappa(t_0) = 0, \varphi(t_0) = \varphi_0, \theta(t_0) = \theta_0, \psi(t_0) = \psi_0, \mathbf{T}(t_0) = \mathbf{T}_0 \ \text{ is taken such that } \ \hat{\mathcal{F}}(\mathbf{T}_0, 0, 0, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) < 0. \tag{50}$$

Remark. If $\sin\theta \neq 0$, then the time derivative of the Euler angles can be explicitly expressed and the following theorem holds.

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Theorem 5. *If $\sin\theta$ is not vanishing during a given history of the deformation gradient, then the evolutions with respect to time of the components of the stress and tensorial hardening variable with respect to the actual orthotropic axes, the Euler angles, ψ, θ and φ, and the scalar hardening variable, κ, are characterized by*

$$\begin{aligned}
\dot{T}_{11} &= -\hat{\mu}(a_{11}\hat{N}^p_{11}(\mathbf{T},\mathbf{A}) + a_{12}\hat{N}^p_{22}(\mathbf{T},\mathbf{A}) + a_{13}\hat{N}^p_{33}(\mathbf{T},\mathbf{A})) + \widetilde{D}_{11}a_{11} + \widetilde{D}_{22}a_{12} + \widetilde{D}_{33}a_{13}\\
\dot{T}_{22} &= -\hat{\mu}(a_{12}\hat{N}^p_{11}(\mathbf{T},\mathbf{A}) + a_{22}\hat{N}^p_{22}(\mathbf{T},\mathbf{A}) + a_{23}\hat{N}^p_{33}(\mathbf{T},\mathbf{A})) + \widetilde{D}_{11}a_{12} + \widetilde{D}_{22}a_{22} + \widetilde{D}_{33}a_{23}\\
\dot{T}_{33} &= -\hat{\mu}(a_{13}\hat{N}^p_{11}(\mathbf{T},\mathbf{A}) + a_{23}\hat{N}^p_{22}(\mathbf{T},\mathbf{A}) + a_{33}\hat{N}^p_{33}(\mathbf{T},\mathbf{A})) + \widetilde{D}_{11}a_{13} + \widetilde{D}_{22}a_{23} + \widetilde{D}_{33}a_{33}\\
\dot{T}_{12} &= -\hat{\mu}a_{44}\hat{N}^p_{12}(\mathbf{T},\mathbf{A}) + \widetilde{D}_{12}a_{44}\\
\dot{T}_{13} &= -\hat{\mu}a_{66}\hat{N}^p_{13}(\mathbf{T},\mathbf{A}) + \widetilde{D}_{13}a_{66}\\
\dot{T}_{23} &= -\hat{\mu}a_{55}\hat{N}^p_{23}(\mathbf{T},\mathbf{A}) + \widetilde{D}_{23}a_{55}\\
\dot{A}_{11} &= \hat{\mu}[(c_0 + 2c_1)\widehat{N}^p_{11}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0 + 2d_1)A_{11}]\\
\dot{A}_{22} &= \hat{\mu}[(c_0 + 2c_2)\widehat{N}^p_{22}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0 + 2d_2)A_{22}]\\
\dot{A}_{33} &= \hat{\mu}[c_0\widehat{N}^p_{33}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)d_0A_{33}]\\
\dot{A}_{12} &= \hat{\mu}[(c_0 + c_1 + c_2)\widehat{N}^p_{12}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0 + d_1 + d_2)A_{12}]\\
\dot{A}_{13} &= \hat{\mu}(c_0 + c_1)\widehat{N}^p_{13}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0 + d_1)A_{13}]\\
\dot{A}_{23} &= \hat{\mu}[(c_0 + c_2)\widehat{N}^p_{23}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0 + d_2)A_{23}]\\
\dot{\varphi} &= \hat{\mu}[\hat{\Omega}^p_{12} - \cot\theta(\hat{\Omega}^p_{13}\cos\varphi + \hat{\Omega}^p_{23}\sin\varphi)] - \widetilde{W}_{12} + \cot\theta(\widetilde{W}_{13}\cos\varphi + \widetilde{W}_{23}\sin\varphi)\\
\dot{\theta} &= \hat{\mu}(\hat{\Omega}^p_{23}\cos\varphi - \hat{\Omega}^p_{13}\sin\varphi) + \widetilde{W}_{13}\sin\varphi - \widetilde{W}_{23}\cos\varphi\\
\dot{\psi} &= \hat{\mu}\frac{1}{\sin\theta}(\hat{\Omega}^p_{13}\cos\varphi + \hat{\Omega}^p_{23}\sin\varphi) - \frac{1}{\sin\theta}(\widetilde{W}_{13}\cos\varphi + \widetilde{W}_{23}\sin\varphi)\\
\dot{\kappa} &= \hat{\mu}\hat{b}
\end{aligned} \tag{51}$$

associated with the yield surface (24), together with (15), namely $\hat{\mu} = \frac{1}{h_c} < \beta > \mathcal{H}(\hat{\mathcal{F}})$ and the initial conditions (50).

Here the components $\widetilde{D}_{ik}$ are expressed in terms of the components of $\mathbf{D} = \{\mathbf{L}\}^s$ in the basis $\{\mathbf{j}_i \otimes \mathbf{j}_k\}$ and of the components $R_{ik} = R_{ik}(\psi,\theta,\varphi)$ of the rotation tensor, and they can be found in (B1).

The expression of the components $\widehat{N}^p_{ij} = \mathbf{m}_i \cdot \hat{N}^p\mathbf{m}_j, \hat{l}_{ij} = \mathbf{m}_i \cdot \hat{\mathbf{l}}\mathbf{m}_j, \hat{\Omega}^p_{ij} = \mathbf{m}_i \cdot \hat{\mathbf{\Omega}}\mathbf{m}_j, i,j = 1,2,3$, are given in (A5), (B2), (C1), (C2) and (C3), respectively. All these functions depend on $\overline{S}_{ij} = T_{ij} - A_{ij}$, but also on A_{ij} since the Armstrong–Frederick hardening law was considered.

The expression of the plastic factor, β, is obtained using $\widehat{N}^p_{ij}$ and $\widetilde{D}_{ij}$ as follows from (A6). Note that function h_c is expressed using $\widehat{N}^p_{ij}$, $\hat{l}_{ij}$ and $\hat{b}$ in the formula (A7).

Remark. If $\sin\theta = 0$ on a certain time interval, then from (45) the equality $\mathbf{m}_3 \cdot (\mathbf{W} - \hat{\mu}\hat{\mathbf{\Omega}}^p)\mathbf{m}_1 = 0, \mathbf{m}_3 \cdot (\mathbf{W} - \hat{\mu}\hat{\mathbf{\Omega}}^p)\mathbf{m}_2 = 0$ necessarily hold. ψ could be considered zero and from (42) all the components $R_{13} = R_{31} = R_{23} = R_{32}$ are vanishing.

6. Rate type models for in-plane rotation and in-plane stress

The particular case of an in-plane rotation of the orthotropy axes and the plane stress state can be derived from the Theorem 4. as it is discussed in the following theorem.

6.1. In-plane rotation of the orthotropy direction

Theorem 6. *Let us consider a deformation process $\mathbf{F} = \mathbf{F}(t), t \in [t_0, t_f]$, with a continuous rate of strain on the time interval $I = [t_0, t_f]$, under the supposition that the shear components $D_{13}(t)$, $D_{23}(t)$, $W_{13}(t)$ and $W_{23}(t)$ are vanish during the deformation process.*

If the initial conditions are given by

$$\begin{aligned}
&\theta(t_0) = 0, \quad \psi(t_0) = 0, \quad \varphi(t_0) = \varphi_0, \quad A_{ij}(t_0) = 0,\ i,j = 1,2,3, \quad \kappa(t_0) = 0,\\
&\mathbf{T}(t_0) = \mathbf{T}_0 \quad \text{such that} \quad T_{13}(t_0) = T_{23}(t_0) = 0,\ \hat{\mathcal{F}}(\mathbf{T}_0, 0, 0, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) < 0.
\end{aligned} \tag{52}$$

then the shear components $T_{13}, T_{23}, A_{13}, A_{23}$ and the Euler angles θ and ψ remain zero during the elasto-plastic process, for every one of the plastic spins considered here, and

$$\begin{aligned}
\dot{T}_{11} &= -\hat{\mu}\,[a_{11}\widehat{N}^p_{11}(\mathbf{T},\mathbf{A}) + a_{12}\widehat{N}^p_{22}(\mathbf{T},\mathbf{A}) + a_{13}\widehat{N}^p_{33}(\mathbf{T},\mathbf{A})] + a_{11}\widetilde{D}_{11}(\varphi) + a_{12}\widetilde{D}_{22}(\varphi) + a_{13}\widetilde{D}_{33}(\varphi)\\
\dot{T}_{22} &= -\hat{\mu}\,[a_{12}\widehat{N}^p_{11}(\mathbf{T},\mathbf{A}) + a_{22}\widehat{N}^p_{22}(\mathbf{T},\mathbf{A}) + a_{23}\widehat{N}^p_{33}(\mathbf{T},\mathbf{A})] + a_{12}\widetilde{D}_{11}(\varphi) + a_{22}\widetilde{D}_{22}(\varphi) + a_{23}\widetilde{D}_{33}(\varphi)\\
\dot{T}_{33} &= -\hat{\mu}\,[a_{13}\widehat{N}^p_{11}(\mathbf{T},\mathbf{A}) + a_{23}\widehat{N}^p_{22}(\mathbf{T},\mathbf{A}) + a_{33}\widehat{N}^p_{33}(\mathbf{T},\mathbf{A})] + a_{13}\widetilde{D}_{11}(\varphi) + a_{23}\widetilde{D}_{22}(\varphi) + a_{33}\widetilde{D}_{33}(\varphi)
\end{aligned}$$

$$\begin{aligned}
\dot{T}_{12} &= -\hat{\mu}\, a_{44} \widehat{N}^p_{12}(\mathbf{T},\mathbf{A}) + a_{44}\widetilde{D}_{12}(\varphi)\\
\dot{A}_{11} &= \hat{\mu}\, [(c_0+2c_1)\widehat{N}^p_{11}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0+2d_1)A_{11}]\\
\dot{A}_{22} &= \hat{\mu}\, [(c_0+2c_2)\widehat{N}^p_{22}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0+2d_2)A_{22}]\\
\dot{A}_{33} &= \hat{\mu}\, [c_0\widehat{N}^p_{33}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)d_0A_{33}]\\
\dot{A}_{12} &= \hat{\mu}\, [(c_0+c_1+c_2)\widehat{N}^p_{12}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A})(d_0+d_1+d_2)A_{12}]\\
\dot{\varphi} &= \hat{\mu}\, \hat{\Omega}^p_{12}(\mathbf{T},\mathbf{A}) - \widetilde{W}_{12}\\
\dot{\kappa} &= \hat{\mu}\hat{b}(\mathbf{T},\mathbf{a},\kappa)
\end{aligned} \tag{53}$$

with the plastic factor given by $\hat{\mu} = \frac{<\beta>}{h_c}\mathcal{H}(\mathcal{F})$.

The proof of the statement can be found in Appendix (B).

Proposition 4. *In the case of in plane rotation of the orthotropic axes, the yield function is reduced to the following expression* $\hat{\mathcal{F}} = (\hat{f}_2)^{3/2} - \gamma\hat{f}_3 - F$,

$$\begin{aligned}
\hat{f}_2 &= K_{11}\overline{S}^2_{11} + K_{22}\overline{S}^2_{22} + K_{33}\overline{S}^2_{33} + K_{m1}\overline{S}^2_{12} + (K_{33}-K_{11}-K_{22})\overline{S}_{11}\overline{S}_{22} + (K_{22}-K_{11}-K_{33})\overline{S}_{11}\overline{S}_{33} + (K_{11}-K_{22}-K_{33})\overline{S}_{22}\overline{S}_{33}\\
\hat{f}_3 &= k_1\overline{S}^3_{11} + k_2\overline{S}^3_{22} + k_3\overline{S}^3_{33} + k_4\overline{S}^2_{11}\overline{S}_{22} + k_5\overline{S}^2_{11}\overline{S}_{33} + k_6\overline{S}^2_{22}\overline{S}_{11} + k_7\overline{S}^2_{22}\overline{S}_{33}\\
&\quad + k_8\overline{S}^2_{33}\overline{S}_{11} + k_9\overline{S}^2_{33}\overline{S}_{22} + k_{10}\overline{S}^2_{12}\overline{S}_{11} + k_{11}\overline{S}^2_{12}\overline{S}_{22} + k_{12}\overline{S}^2_{12}\overline{S}_{33} + k_{19}\overline{S}_{11}\overline{S}_{22}\overline{S}_{33}.
\end{aligned} \tag{54}$$

Remark. In our representation of the yield function only ten independent yield constants, denoted by B_k, appear in the expression of f_3, in contrast to expressions presented by Cazacu and Barlat (2004) where eleven independent coefficients appear.

In conclusion:

- The in-plane elastic rotation tensor is characterized by non-zero components

$$R_{11} = R_{22} = \cos\varphi, \quad R_{12} = -\sin\varphi, \quad R_{21} = \sin\varphi, \quad R_{33} = 1, \tag{55}$$

- Functions $(\widehat{N}^p)_{ij}$, with $\widehat{N}^p_{13} = \widehat{N}^p_{23} = 0$, are given in (A5), while functions β and h_c depend on T_{ij}, A_{ij}, $\widetilde{D}_{ij}$ and φ as can be seen from (A6) and (A7) with the following vanishing components $\widetilde{D}_{13} = \widetilde{D}_{23} = 0$.
- There is a single non-vanishing component of all plastic spins considered herein, namely $\hat{\Omega}^p_{12}$, whose expression is given by

$$\begin{aligned}
&\hat{\Omega}^p_{12} = \overline{A}_1\overline{S}_{12} + \overline{A}_2(\overline{S}_{11}+\overline{S}_{22})\overline{S}_{12} \quad \text{generated by } \mathbf{S},\\
&\hat{\Omega}^p_{12} = \bar{\eta}_1\widehat{N}^p_{12} + \bar{\eta}_2(\widehat{N}^p_{11}+\widehat{N}^p_{22})\widehat{N}^p_{12} \quad \text{generated by } \mathbf{N^p},\\
&\hat{\Omega}^p_{12} = \hat{\eta}_1(\overline{S}_{11}\widehat{N}^p_{12} + \overline{S}_{12}\widehat{N}^p_{22}) + \hat{\eta}_2(\overline{S}_{12}\widehat{N}^p_{11} + \overline{S}_{22}\widehat{N}^p_{12}) \quad \text{generated by } \mathbf{S} \text{ and } \mathbf{N^p}.
\end{aligned} \tag{56}$$

- We add the expression for the constitutive function $\hat{b}$, derived from (41) or (43).

6.2. In-plane stress state

The stress state is plane during the deformation process with respect to the axes $\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3$, if and only if

$$\mathbf{j}_1\cdot\mathbf{T}\mathbf{j}_3 = \mathbf{j}_2\cdot\mathbf{T}\mathbf{j}_3 = \mathbf{j}_3\cdot\mathbf{T}\mathbf{j}_3 = 0. \tag{57}$$

Let us suppose a deformation process characterized by the zero shear components $D_{13} = D_{23} = 0$, and in-plane rotation of the orthotropy directions. The orthotropic axis, $\mathbf{m}_3 = \mathbf{j}_3$, remains fixed during the process, and $\mathbf{j}_3\cdot\mathbf{T}\mathbf{j}_3 = \mathbf{m}_3\cdot\mathbf{T}\mathbf{m}_3 = 0$.

We investigate the necessary condition to have the plane stress associated with a rotation in the plane of the orthotropy directions. In order to understand what is in-plane stress state means, we assert the following important **remark**:

Remark. To have $T_{33} = 0$ during the process, the component of the stretching along the direction perpendicular to the plane, $\widetilde{D}_{33}$, should be derived from (53) by the following equation

$$\widetilde{D}_{33} = \frac{\hat{\mu}}{a_{33}}(a_{13}\hat{N}^p_{11} + a_{23}\hat{N}^p_{22} + a_{33}\hat{N}^p_{33}) - \frac{a_{13}}{a_{33}}\widetilde{D}_{11} - \frac{a_{23}}{a_{33}}\widetilde{D}_{22}, \tag{58}$$

in which the plastic factor on the yield surface is given by $\hat{\mu} = \frac{<\beta>}{h_c}$, where β is depending on $\widetilde{D}_{33}$, as it can be seen from the formula (15).

We conclude that $\widetilde{D}_{33}$ can not be arbitrarily given in the considered deformation process, and it results to be dependent on $\widetilde{D}_{11}, \widetilde{D}_{22}, \widetilde{D}_{12}$. Consequently, in plane stress state, the component $\widetilde{D}_{33}$ is not necessarily vanishing, despite the case considered for instance in Han et al. (2002). As a direct consequence of the previously given remark and using the same arguments as in Cleja-Ţigoiu (2007) we give the following theorem, which characterizes the plane stress state.

Theorem 7. *Let us consider a deformation process with* $D_{13} = D_{23} = W_{13} = W_{23} = 0$ *and the orthotropy direction* $\mathbf{n}_3$ *is perpendicular to the plane* $(\mathbf{j}_1, \mathbf{j}_2)$.

1. The differential system which describes, *in the plane stress process*, the material response is described by

$$\begin{aligned}
\dot{T}_{11} &= -\hat{\mu}_{pl}[\tilde{a}_{11}\,\hat{N}^p_{11}(\mathbf{T},\mathbf{A}) + \tilde{a}_{12}\hat{N}^p_{22}(\mathbf{T},\mathbf{A})] + \tilde{a}_{11}\,\widetilde{D}_{11}(\varphi) + \tilde{a}_{12}\,\widetilde{D}_{22}(\varphi)\\
\dot{T}_{22} &= -\hat{\mu}_{pl}[\tilde{a}_{12}\,\hat{N}^p_{11}(\mathbf{T},\mathbf{A}) + \tilde{a}_{22}\,\hat{N}^p_{22}(\mathbf{T},\mathbf{A})] + \tilde{a}_{12}\,\widetilde{D}_{11}(\varphi) + \tilde{a}_{22}\widetilde{D}_{22}(\varphi)\\
\dot{T}_{12} &= -\hat{\mu}_{pl}a_{44}\widehat{N}^p_{12}(\mathbf{T},\mathbf{A}) + a_{44}\widetilde{D}_{12}(\varphi)\\
\dot{A}_{11} &= \hat{\mu}_{pl}[(c_0+2c_1)\widehat{N}^p_{11}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0+2d_1)A_{11}]\\
\dot{A}_{22} &= \hat{\mu}_{pl}[(c_0+2c_2)\widehat{N}^p_{22}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0+2d_2)A_{22}]\\
\dot{A}_{12} &= \hat{\mu}_{pl}[(c_0+c_1+c_2)\widehat{N}^p_{12}(\mathbf{T},\mathbf{A}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)(d_0+d_1+d_2)A_{12}]\\
\dot{\kappa} &= \hat{\mu}_{pl}\hat{b}(\mathbf{T},\mathbf{A},\kappa)\\
\dot{\varphi} &= \hat{\mu}_{pl}\hat{\Omega}^p_{12}(\mathbf{T},\mathbf{A}) - \widetilde{W}_{12}(\varphi)
\end{aligned} \tag{59}$$

with the hardening constant $c_0 = 0$ and with the initial conditions written in (52) at which we add $T_{33} = 0$.

2. The expression of the modified plastic factor, denoted by $\hat{\mu}_{pl}$, is derived under the form

$$\begin{aligned}
\hat{\mu}_{pl} &= \frac{\beta_{pl}}{h_{c,pl}},\\
\beta_{pl} &= [\tilde{a}_{11}\,\hat{N}^p_{11} + \tilde{a}_{12}\,\hat{N}^p_{22}]\widetilde{D}_{11} + [\tilde{a}_{12}\,\hat{N}^p_{11} + \tilde{a}_{22}\,\hat{N}^p_{22}]\widetilde{D}_{22} + 2a_{44}\hat{N}^p_{12}\widetilde{D}_{12},
\end{aligned} \tag{60}$$

if the modified hardening parameter $\hat{h}_{c,pl}$ is positive, where the expression for the hardening parameter is written in the following form

$$h_{c,pl} = \widehat{N}^p_{11}(\tilde{a}_{11}\widehat{N}^p_{11} + \tilde{a}_{12}\widehat{N}^p_{22}) + \widehat{N}^p_{22}(\tilde{a}_{12}\widehat{N}^p_{11} + \tilde{a}_{22}\widehat{N}^p_{22}) + 2a_{44}(\widehat{N}^p_{12})^2 + \widehat{N}^p_{11}\,\hat{l}_{11} + \widehat{N}^p_{22}\,\hat{l}_{22} + 2\widehat{N}^p_{12}\,\hat{l}_{12} + (\partial_\kappa F(\kappa))\,\hat{b}. \tag{61}$$

3. Here the set of reduced elastic coefficients are defined by the combination of the elastic moduli which is realized during the deformation process in-plane stress state

$$\tilde{a}_{11} = a_{11} - \frac{a_{13}^2}{a_{33}}, \quad \tilde{a}_{12} = a_{12} - \frac{a_{13}a_{23}}{a_{33}}, \quad \tilde{a}_{22} = a_{22} - \frac{a_{23}^2}{a_{33}}. \tag{62}$$

4. The axial stretching is given by

$$\widetilde{D}_{33}(\varphi) = \frac{\hat{\mu}_{pl}}{a_{33}}(a_{13}\hat{N}^p_{11}(\mathbf{T},\mathbf{a}) + a_{23}\hat{N}^p_{22}(\mathbf{T},\mathbf{a}) + a_{33}\hat{N}^p_{33}(\mathbf{T},\mathbf{a})) - \frac{a_{13}}{a_{33}}\widetilde{D}_{11}(\varphi) - \frac{a_{23}}{a_{33}}\widetilde{D}_{22}(\varphi). \tag{63}$$

Proof. By eliminating the component $\widetilde{D}_{33}$ from (58) together with the consistency condition written in terms of $\hat{\mu}$, namely if $\hat{\mu} > 0$ then $\dot{\hat{\mathcal{F}}} \equiv \beta - \hat{\mu}h_c = 0$, we obtain the following expression

$$\hat{\mu}\left[h_c - \frac{1}{a_{33}}(a_{13}\hat{N}^p_{11} + a_{23}\hat{N}^p_{22} + a_{33}\hat{N}^p_{33})^2\right] = [\tilde{a}_{11}\,\hat{N}^p_{11} + \tilde{a}_{12}\,\hat{N}^p_{22}]\widetilde{D}_{11} + [\tilde{a}_{12}\,\hat{N}^p_{11} + \tilde{a}_{22}\hat{N}^p_{22}]\widetilde{D}_{22} + 2a_{44}\hat{N}^p_{12}\widetilde{D}_{12}. \tag{64}$$

Just the expression in the large bracket defines the modified hardening parameter $\hat{h}_{c,pl}$, those expression is written in (61). □

The component A_{33} is not vanishing as a direct consequence of the fact that $\hat{N}^p_{33}$ is not vanishing in this case. The evolution in time of this component is given by an appropriate differential equation derived from (53), namely $\dot{A}_{33} = \hat{\mu}_{pl}[c_0\widehat{N}^p_{33}(\mathbf{T},\mathbf{a}) - \hat{b}(\mathbf{T},\mathbf{A},\kappa)d_0A_{33}]$. A_{33} is vanishing along the solution of the differential equation if and only if the constant that appears in the evolution equation for the tensorial hardening parameter vanishes, i.e., $c_0 = 0$.

Comments. In the model proposed herein, the component of the back-stress, A_{33}, has an evolution related to the evolution of the plastic strain in the normal direction, in contrast to the model adopted by Cleja-Ţigoiu (2007) that leads to $A_{33} = 0$. In order to avoid $A_{33} \neq 0$, we introduce the limitation of the model to the case of Armstrong–Frederick hardening law with $c_0 = 0$. This limitation is not necessary if no supposition concerning the stress state has been made. Hahm and Kim (2008) noticed that for sheet materials in a plane stress state, the back stress measurement in the thickness direction, in our notation A_{33}, remains unknown. We adopted a similar point of view to have $A_{33} = 0$ as was made by Truoung Qui and Lippmann (2001) and Hahm and Kim (2008) when the material is subject to a plane stress.

Proposition 5. *In the case of a plane stress state, the yield function (50) is reduced to the following expression* $\hat{\mathcal{F}} = (\hat{f}_2)^{3/2} - \gamma\hat{f}_3 - F$, *for* $\gamma = 1$, *and where*

$$\begin{aligned}\hat{f}_2 &= K_{11}\overline{S}_{11}^2 + K_{22}\overline{S}_{22}^2 + K_{33}\overline{S}_{33}^2 + K_{m1}\overline{S}_{12}^2 + (K_{33} - K_{11} - K_{22})\overline{S}_{11}\overline{S}_{22},\\ \hat{f}_3 &= k_1\overline{S}_{11}^3 + k_2\overline{S}_{22}^3 + k_4\overline{S}_{11}^2\overline{S}_{22} + k_6\overline{S}_{22}^2\overline{S}_{11} + k_{10}\overline{S}_{12}^2\overline{S}_{11} + k_{11}\overline{S}_{12}^2\overline{S}_{22},\end{aligned} \tag{65}$$

with the six coefficients k_i, $i \in \{1,2,4,6,10,11\}$ given in (A2), (A3), and the function $F = F(\kappa)$ describes the isotropic hardening.

7. Numerical simulations

First we determine the initial yield surface, and the material constants involved in our model, which are consistent with experimental data, and second the model will be applied to simulate the material behaviour in certain test experiments.

7.1. Initial yield surface for orthotropic material: set of material parameters compatible with experimental data

We refer to the initial yield surface, which is described by function $\hat{\mathcal{F}}$, together with functions $\hat{f}_2$ and $\hat{f}_3$ given by (28), (31) and (33), respectively, when $\mathbf{P} = \mathbf{I}, \mathbf{R}^e = \mathbf{I}, \gamma = 1$, namely

$$\hat{\mathcal{F}}(\mathbf{T}, \mathbf{A} = \mathbf{0}, \kappa = 0, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) := (\hat{f}_2)^{3/2} - \gamma\hat{f}_3 - \sigma_Y^3 = 0.$$

In order to define the material yield constants, consistent experimental data, from the plane stress state experiment, which can be found in Verma et al. (2011), will be employed. The relationships between the yield constants denoted by

$$\xi = (K_{11}, K_{22}, K_{33}, K_{m1}, \{B_k\}_{1 \leqslant k \leqslant 10}) \equiv (\xi_i)_{1 \leqslant i \leqslant 14} \tag{66}$$

and the experimental data, are obtained for the uniaxial stress state which is applied along a direction, say $\boldsymbol{v}$, and for the equi-biaxial stress, respectively,

$$\mathbf{T} = \mathbf{T}|_\alpha := \tilde{\sigma}_\alpha \boldsymbol{v} \otimes \boldsymbol{v}, \quad \mathbf{T} = \mathbf{T}|_b = \tilde{\sigma}_b(\mathbf{n}_1 \otimes \mathbf{n}_1 + \mathbf{n}_2 \otimes \mathbf{n}_2), \tag{67}$$

where $\boldsymbol{v} = \cos\alpha\,\mathbf{n}_1 + \sin\alpha\,\mathbf{n}_2$ is in the plane $(\mathbf{n}_1, \mathbf{n}_2)$, and $\alpha \in [0, \pi/2]$.

The coefficient of plastic orthotropy is associated with direction $\boldsymbol{v}$

$$r_\alpha := \frac{D^p_{\alpha+90}|_\alpha}{D^p_{33}|_\alpha} = -\frac{D^p_{\alpha+90}|_\alpha}{D^p_{11}|_\alpha + D^p_{22}|_\alpha}, \tag{68}$$

here $D^p_{\alpha+90}|_\alpha = \boldsymbol{v}_\perp \cdot \mathbf{D}^p\, \boldsymbol{v}_\perp$, $D^p_{33}|_\alpha = \mathbf{n}_3 \cdot \mathbf{D}^p \mathbf{n}_3$, where $\boldsymbol{v}_\perp$ denotes the direction in the plane perpendicular to $\boldsymbol{v}$. The rates of plastic strains involved in (68) have to be associated with stress states that reached the initial yield surface.

We give the **expressions for the plastic orthotropic parameters** previously introduced, in terms of the material constants which enter the yield surface, by eliminating the components of the stresses. We take into account that the considered stress state lies on the initial yield surface. Note that the material constants K_{m2}, K_{m3} do not enter the expressions calculated for the plastic orthotropic parameters as the stress components T_{33}, T_{13}, T_{23} are zero in stress plane experiments.

Let us introduce

$$\sigma(\xi, \alpha) = \sigma_\alpha, \quad \sigma^c(\xi, \alpha) = \sigma^c_\alpha, \quad \sigma_b(\xi) = \sigma_b, \quad \sigma^c_b(\xi) = \sigma^c_b, \quad r(\xi, \alpha) = r_\alpha, \tag{69}$$

for the normalized yield stresses in tension $\sigma_\alpha = \frac{\tilde{\sigma}_\alpha}{\sigma_Y}$, $\sigma_b = \frac{\tilde{\sigma}_b}{\sigma_Y}$, and in compression σ^c_α, σ^c_b, respectively.

The dimensionless uniaxial yield stress are calculated for the stress state (66) using the formula

$$\sigma(\xi, \alpha) = \{(f_2|_\alpha)^{3/2} - \gamma\,(f_3|_\alpha)\}^{-1/3}, \ \sigma^c(\xi, \alpha) = \{(f_2|_\alpha)^{3/2} + \gamma\,(f_3|_\alpha)\}^{-1/3} \tag{70}$$

where

$$\begin{aligned} f_2|_\alpha &= K_{11}\cos^4\alpha + K_{22}\sin^4\alpha + (K_{33} - K_{11} - K_{22} + K_{m1})\cos^2\alpha\sin^2\alpha, f_3|_\alpha \\ &= k_1(\xi)\cos^6\alpha + k_2(\xi)\sin^6\alpha + (k_4(\xi) + k_{10}(\xi))\cos^4\alpha\sin^2\alpha + (k_6(\xi) + k_{11}(\xi))\sin^4\alpha\cos^2\alpha. \end{aligned} \tag{71}$$

The dimensionless equi-biaxial yield stress are calculated as follows:

$$\begin{aligned}\sigma_b(\xi) &= \frac{1}{[(K_{33})^{3/2} - \gamma(k_1 + k_2 + k_4 + k_6)(\xi)]^{1/3}}, \quad K_{33} = \xi_3\\ \sigma^c_b(\xi) &= \frac{1}{[(K_{33})^{3/2} + \gamma(k_1 + k_2 + k_4 + k_6)(\xi)]^{1/3}}.\end{aligned} \tag{72}$$

The coefficients of anisotropy (68) are calculated in terms of the appropriate components of the plastic stretching, but for the stress state (67)

$$r(\xi,\alpha) = -\frac{E_1}{E_2} \tag{73}$$

with E_1 and E_2 calculated in terms of α and ξ, and explicitly given in Appendix (D1).

Further we investigate the following problem: find the set of constants ξ such that the experimental values of the parameters which characterize the orthotropy are satisfied.

We write the conditions which describe the restrictions (32) on the physical parameters

$$C = \{\xi \in \mathbf{R}^{14} | \xi_1 + \xi_2 - \xi_3 > 0,\ \xi_1 + \xi_3 - \xi_2 > 0,\quad \xi_2 + \xi_3 - \xi_1 > 0,\ \xi_4 > 0\}, \tag{74}$$

and we define the admissible set

$$D = \{\xi \in \mathbf{R}^{14} |\ \forall\ \alpha \in \alpha_I \quad \exists\ \sigma(\xi,\alpha), \sigma^c(\xi,\alpha), \sigma_b(\xi), \sigma_b^c(\xi), r(\xi,\alpha=0),\ \alpha \in \alpha_I\}. \tag{75}$$

with $\alpha_I = \{0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ\}$.

Remark. The set C is non-empty and convex and D is an open set.

For any $\xi \in C \cap D$ the set $F^{th}(\xi)$ is defined by

$$F^{th}(\xi) = (\sigma(\xi,\alpha), \sigma^c(\xi,\alpha), \sigma_b(\xi), \sigma_b^c(\xi), r(\xi,\alpha=0),\ \alpha \in \alpha_I) \in R^{17}. \tag{76}$$

The set of experimental data given in Verma et al. (2011) for an ultra low-carbon interstitial free high strength steel, were gathered together in the Table 1.

The normalized values for yield limits are written in Table 1. The experimental data given in Verma et al. (2011) contain three coefficients of plastic orthotropy, which are determined for the engineering strains between 0.05 and 0.15. We used only $r^{exp}_{\alpha=0}$, which is measured for the initial yield surface. We introduce the set of experimental data, F^{exp}, which contain the data from the Table 1, at which we added another ten values from the experimental graphics given in the mentioned paper, namely

$\sigma^f_{\alpha=15}, \sigma^f_{\alpha=30}, \sigma^f_{\alpha=60}, \sigma^f_{\alpha=75}$ in traction, and

$\sigma^{c,f}_{\alpha=15}, \sigma^{c,f}_{\alpha=30}, \sigma^{c,f}_{\alpha=45}, \sigma^{c,f}_{\alpha=60}, \sigma^{c,f}_{\alpha=75}$ in compression and $\sigma^{c,f}_b = \sigma^{exp}_b$.

The following problem arises: find $\xi \in C \cap D$ such that $F^{th}(\xi) = F^{exp}$. This algebraic system is over determined and consequently a minimization procedure has been applied to find the numerical values of the yield material parameters.

Following the idea of Nixon et al. (2010) and Banabic et al. (2003), in order to find the material constants compatible with the experimental data in the sense formalized in (75), we introduce the function $\bar{f} : \mathbf{R}^{14} \to \mathbf{R}$ defined to be equivalent to the Euclidean distance between $F^{th}(\xi)$ and F^{exp} in R^{17}

$$\bar{f}(\xi) = \sum_{i=1}^{17} w_i (F^{th}(\xi)_i - F_i^{exp})^2 \tag{77}$$

with $w_i > 0\ \forall\ i \in \{1, \dots 17\}$. In order to obtain better results for $\sigma_\alpha, \alpha \in \{0^\circ, 45^\circ, 90^\circ\}$, $\sigma^c_\alpha, \alpha \in \{0^\circ, 90^\circ\}$, σ_b, $r_{\alpha=0}$, than for the other, we use $w_1 = w_4 = w_7 = w_8 = w_{14} = w_{15} = 1000$, $w_{17} = 10$ and $w_2 = w_3 = w_5 = w_6 = w_9 = w_{10} = w_{11} = w_{12} = w_{13} = w_{16} = 1$.

The following set of the yield dimensionless constants is obtained:

$$\begin{aligned} &K_{11} = 1.0050,\ K_{22} = 0.9793,\ K_{33} = 0.9350,\ K_{m1} = 3.0760, \\ &B_1 = -0.8802,\ B_2 = 0.5506,\ B_3 = 7.6231,\ B_4 = 0.1304,\ B_5 = -6.9450, \\ &B_6 = -1.5898,\ B_7 = 1,\ B_8 = B_9 = B_{10} = 0. \end{aligned} \tag{78}$$

Remarks.

1. We mention that the indirect experimental data analyzed herein are emphasized in connection with certain anisotropy parameters and not with the effective measurements of the stress state along certain paths as was done, for instance, in the papers by Phillips and Kasper (1973) and Phillips and Liu (1972).
2. In the case when $\gamma = 0$, i.e., for a quadratical yield surface, the same procedure is applied and it is obtained higher value $||F^{th} - F^{exp}||$ for the Euclidean norm. Consequently, by increasing the number of numerical parameters to be defined, the efficiency of the proposed method is increased. Hence the case when $\gamma = 1$ allows us to build a better approximation for the experimental data.

Table 1
Experimental data given in Verma et al. (2011).

Orthotropic parameters	$\sigma^{exp}_{\alpha=0}$	$\sigma^{exp}_{\alpha=45}$	$\sigma^{exp}_{\alpha=90}$	$\sigma^{c,exp}_{\alpha=0}$	$\sigma^{c,exp}_{\alpha=90}$	σ^{exp}_b	$r^{exp}_{\alpha=0}$
Exp. values	1	193/196	192/196	195/196	205/196	195/196	1.59

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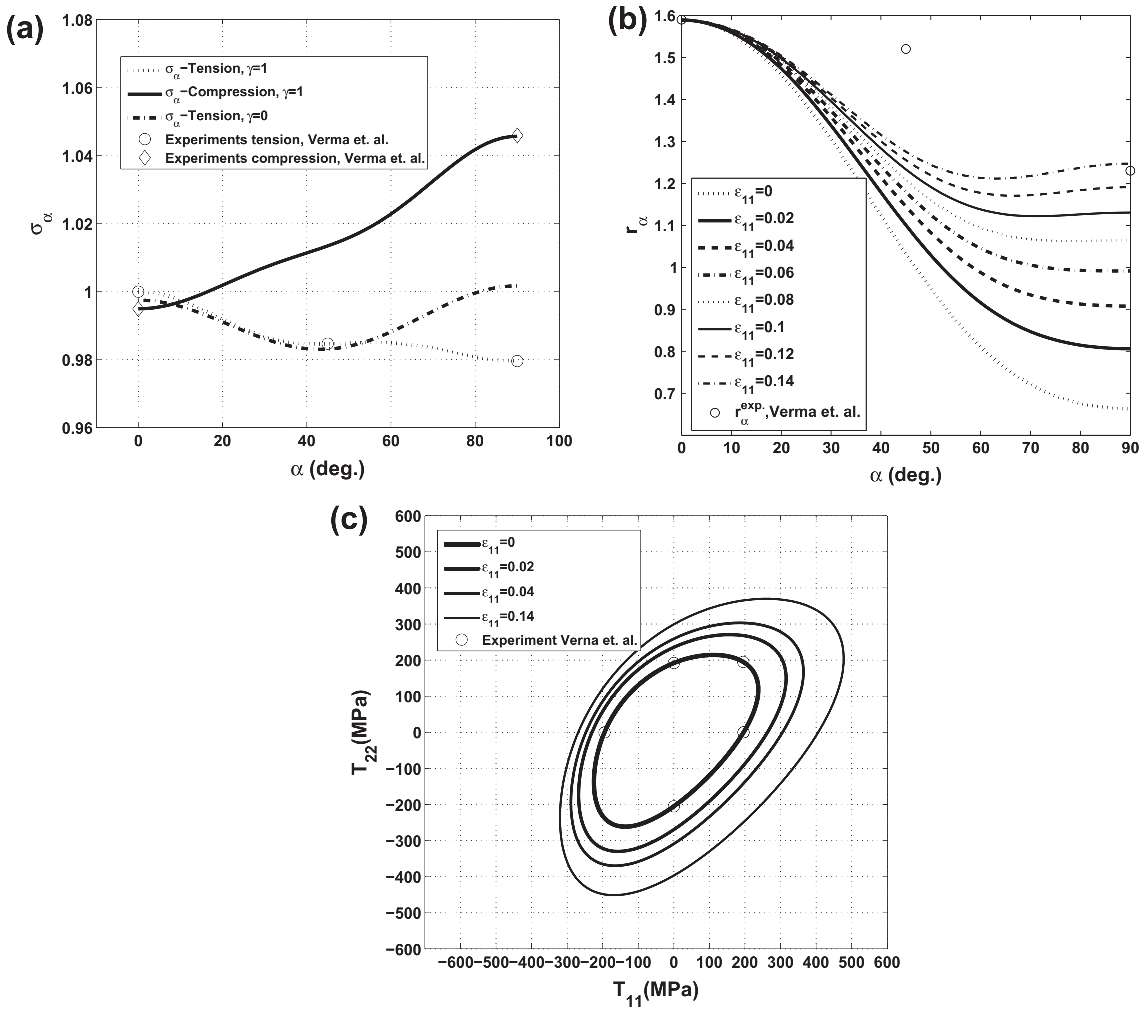

Fig. 2. (a) Uniaxial yield stress σ_α and experimental data. (b) Anisotropy coefficient r_α and (c) yield stress curves (T_{11}, T_{22}) plotted for various values of the uniaxial strain ε_{11}.

3. A good approximation of the experimental data can be ensured for σ_α as it can be seen from Fig. 2a, in contrast with the graphs for the orthotropy coefficients $r_\alpha(\xi, \varphi)$, which are plotted in Fig. 2b. This is a direct consequence of the fact that the initial yield function has been calibrated using the set of yield stresses. We pointed out that the orthotropy coefficients $r_\alpha(\xi, \varphi)$ have been determined by Verma et al. (2011) for the engineering strains between 0.05 and 0.15, while in the theoretical formulae we introduced the hypothesis $\mathbf{P} = I$, i.e., $\varepsilon^p = 0$. In Fig. 2c the yield stress curves (T_{11}, T_{22}) derived from the yield surface are plotted for various values of deformation and they are compared with the experimental yield points. The tension–compression asymmetry is an important subject for metal modelling (see Nixon et al. (2010), Kuroda (2003)) and the model proposed here can describe this phenomenon.

7.2. Determination of the hardening parameters

Note that just the material constants c_1, c_2 and d_1, d_2 characterize the influence of the orthotropy on the hardening. The hardening parameters which enter the differential system (59) with $c_0 = 0$ are determined to be compatible with experimental data given in Verma et al. (2011). To have similar conditions with the performed experiments, which means that the change in the orthotropy directions is not occurred in the process, we consider no plastic spins and at the initial moment t_0, $A_{ij}(t_0) = 0, \kappa(t_0) = 0$. Consequently the time derivative at *time* t_0, for the normal components of the stress and back-stress can be derived from the differential system. Only the kinematic hardening parameters c_1, d_1 enter the appropriate expressions for normal components. We mention that only when the the scalar internal variable is considered to characterizes the plastic work, and the associated scalar hardening function $F = F(\kappa)$ is considered to be of Voce type, see (43), our model is able to describe with accuracy the tension–compression tests. Since $\dot{\kappa}(t_0) \neq 0$, we can consider κ to be locally a new

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independent variable, and by dividing the equations by $\dot{\kappa}$ no presence of $\mu(t_0)$ is involved now. The new form of the initial variation of the normal components have been derived from the simulated uniaxial test. It follows that $c_1 = \frac{T_{11}}{2\sigma_Y}\frac{dA_{11}}{d\kappa}$ cal cal.

The dimensionless variable x_c is taken from Verma et al. (2011) and y_c from the condition to have $F(\kappa_0) = \sigma_Y^3$, which means $x_c + y_c = 1$, namely $x_c = 394.66/196, y_c = -1.0136$.

For the determination of the kinematic hardening variable d_1 we used again the differential system (59) written for the uniaxial monotonic tension and the Fig. 8 from Verma et al. (2011). In order to find the pairs of values ($T_{11} = 300$ MPa, $\varepsilon_{11} = 0.004$) and ($T_{11} = 380$ MPa, $\varepsilon_{11} = 0.1$) the following numerical values have to be considered $d_1 = 15, z_c = 22$.

The *kinematic hardening variables* d_0, c_2, d_2 *remain to be determined*. Let us remark the component A_{22} could influence the behaviour of the material. At a first step we consider d_0, c_2, d_2 are vanishing parameters, further on in the case of a plane stress state we simulate the experiments used by Kim and Yin (1997), using the plastic spin III. The presence of the non vanishing kinematic constants, say $c_2 = 120, d_2 = 250$, leads to a better approximation of the experimental data than the previously considered $c_2 = d_2 = 0$, as it can be seen in Fig. 4a.

7.3. *Numerical simulations for the homogeneous deformation of a sheet*

We consider a sheet made up from an orthotropic elasto-plastic material with the edges parallel to the fixed axes $\mathbf{j}_k, k = 1, 2, 3$, and having the initial orthotropic axes, $\mathbf{n}_k, k = 1, 2, 3$, generally with an orientation different from the fixed axes, see Fig. 1. The plate is subjected to one of the following homogeneous deformations processes:

(1) The axial deformations

$$\mathbf{F} = \lambda_1(t)\mathbf{j}_1 \otimes \mathbf{j}_1 + \lambda_2(t)\mathbf{j}_2 \otimes \mathbf{j}_2 + \lambda_3(t)\mathbf{j}_3 \otimes \mathbf{j}_3, \tag{79}$$

with the strains $\lambda_1, \lambda_2, \lambda_3 : [t_0, t_f) \to \mathbf{R}$ such that $\lambda_j(t_0) = 1$. Thus

$$\mathbf{L} = \sum_{i=1}^{i=3} \frac{\dot{\lambda}_i}{\lambda_i}\mathbf{j}_i \otimes \mathbf{j}_i, \quad \mathbf{D} = \mathbf{L}, \quad \mathbf{W} = \mathbf{0}. \tag{80}$$

No motion spin is associated with this homogeneous deformation process and consequently no rotation of the material element is produced, while the otroptropy axes are changing if the non-vanishing spin is involved in the model.

(2) The shear deformation in plane $\mathbf{j}_1, \mathbf{j}_2$ with normal strain

$$\mathbf{F} = \sum_{i=1}^{2}\mathbf{j}_i \otimes \mathbf{j}_i + \Gamma(t)\mathbf{j}_1 \otimes \mathbf{j}_2 + \lambda_3(t)\mathbf{j}_3 \otimes \mathbf{j}_3, \tag{81}$$

with $\Gamma, \lambda_3 : [t_0, t_f) \to \mathbf{R}$ such that $\Gamma(t_0) = 0$ and $\lambda_3(t_0) = 1$. The symmetric part of the velocity gradient and the total spin which is nonzero are given by

$$\mathbf{D} = \frac{1}{2}\dot{\Gamma}(\mathbf{j}_1 \otimes \mathbf{j}_2 + \mathbf{j}_2 \otimes \mathbf{j}_1) + \frac{\dot{\lambda}_3}{\lambda_3}\mathbf{j}_3 \otimes \mathbf{j}_3, \quad \mathbf{W} = \frac{1}{2}\dot{\Gamma}(\mathbf{j}_1 \otimes \mathbf{j}_2 - \mathbf{j}_2 \otimes \mathbf{j}_1). \tag{82}$$

(3) The plane stress state has also been considered.

For the cases considered, the corresponding differential systems are numerically integrated using a Matlab code following Hanselman and Littlefield (1997) and Moler (2011).

The graphs for the components of the tensorial fields have been plotted with respect to the geometrical fixed axes $\mathbf{j}_i \otimes \mathbf{j}_k$..

In order to be as close as possible to the experiments of Kim and Yin (1997), we use the elastic constants of a low carbon steel with a cubic symmetry.

The elastic constants have been divided by $\sigma_Y = 196$ MPa:

$a_{11} = 1404.049, a_{12} = 5734.85, a_{44} = 4183.67$, $a_{22} = a_{33} = a_{11}, a_{13} = a_{23} = a_{12}, a_{55} = a_{66} = a_{44}$, and they correspond to the Poisson's ratio $\nu = 0.29$, shear modulus $\mu = 82$ GPa and Young's modulus $E = 210$ GPa using (A3), see also http://www.makeitfrom.com.

The yield constants: are given in (79) for plane rotation of the orthotropy axes, and for the general rotation $K_{m2} = K_{m3} = 2K_{m1}$ have to be added.

The hardening constants:

– for kinematic hardening: $c_0 = 10,\ c_1 = 2.65,\ c_2 = 120,\ d_0 = 5,\ d_1 = 15,\ d_2 = 250$;

– for scalar hardening: $x_c = 2.0136,\ y_c = -1.0136,\ z_c = 22$, (Voce-type).

The plastic spin constants:

– for the plastic spin generated by both $\overline{\mathbf{S}}$ and $\hat{\mathbf{N}}^p$: $\tilde{\eta} = 5102,\ \tilde{\eta}_1 = 2118,\ \tilde{\eta}_2 = 900,\ \tilde{\eta}_3 = -500$..

All the material parameters used in application can be found in the Tables 2 and 3.

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Table 2
Non-dimensionalized material parameters: elastic constants (see Eq. (A4)), yield constants (see Eqs. (65)), hardening constants (see Eqs. (38) and (43)).

Elastic constants (divided by σ_Y)	Yield constants K_{ij}	Yield constants B_k	Kinematic hardening (c_0, c_1, c_2 divided by σ_Y)	Scalar hardening
$a_{11} = 1404.049$	$K_{11} = 1.0050$	$B_1 = -0.8802$	$c_0 = 10$	$x_c = 2.0136$
$a_{12} = 5734.85$	$K_{22} = 0.9793$	$B_2 = 0.5506$	$c_1 = 2.65$	$y_c = -1.0136$
$a_{44} = 4183.67$	$K_{33} = 0.9350$	$B_3 = 7.6231$	$c_2 = 120$	$z_c = 22$
$a_{22} = a_{33} = a_{11}$	$K_{m1} = 3.0760$	$B_4 = 0.1304$	$d_0 = 5$	
$a_{13} = a_{23} = a_{12}$	$K_{m2} = 6.1520$	$B_5 = -6.9450$	$d_1 = 15$	
$a_{55} = a_{66} = a_{44}$	$K_{m3} = 6.1520$	$B_6 = -1.5898$	$d_2 = 250$	
		$B_7 = 1$		
		$B_8 = B_9 = B_{10} = 0$		

Table 3
Material parameters corresponding to the plastic spins (see Eqs. (56)).

Plastic spin I ($\overline{A}_2$ divided by σ_Y)	Plastic spin II ($\overline{\eta}_2$ multiplied by σ_Y^2)	Plastic spin III (multiplied by σ_Y)
$\overline{A}_1 = 0$	$\overline{\eta}_1 = 0$	$\hat{\eta}_1 = 6502$
$\overline{A}_2 = -180$	$\overline{\eta}_2 = -10$	$\hat{\eta}_2 = 7220$

Evolution of the yield surface. In the numerical simulations, the deformation process (79) is considered with a non-decreasing λ_1 applied along the axis $\mathbf{j}_1$, namely $\dot{\lambda}_1 \geqslant 0$ during the process, then a new time variable $x = \ln(\lambda_1)$ could be introduced by a change of variables $\frac{dx}{dt} = \frac{\dot{\lambda}_1(t)}{\lambda_1(t)}$.

The differential system describing the evolution in time of $\mathbf{T}$, $\mathbf{A}$ and the scalar hardening variable, κ, is given with the unknown functions expressed in terms of $x = \ln(\lambda_1)$, and $\varepsilon_{11} = \frac{1}{2}(\lambda_1^2 - 1)$. We choose four different values $\varepsilon_{11} \in \{0, 0.02, 0.04, 0.14\}$ reached during the process. By integrating the differential system up to a fixed ε_{11}, the appropriate values of the tensorial and scalar hardening variables could be numerically calculated for the considered strain. When the appropriate hardening variables are replaced in the yield expression $\hat{\mathcal{F}} = 0$ with the yield function (65) written for the plane stress state, the projections of this current yield surface on the (T_{11}, T_{22})−plane, i.e., for $T_{12} = T_{13} = T_{23} = T_{33} = 0$, could be derived and these have been plotted in Fig. 2c. A strong differential effect can be noticed from the intersections of the curves (T_{11}, T_{22}) with the coordinate axes. This phenomenon is caused by the evolution of the kinematic hardening variable.

Let us remark that only under the hypothesis that the anisotropy axes remain fixed during the deformation process such a representation is correct since the axes are directly involved in the expression of the yield surface.

The uniaxial stress state case is considered for the deformation process (79) in order to simulate both of processes Tension–Compression–Tension (TCT) and Compression–Tension–Compression (CTC). The initial orthotropy axes are parallel with the edge of the sheet and they remain fixed, since no plastic spin is involved in this process. The strain–stress curves are plotted in Fig. 3, for the following pre-strain $\varepsilon^{pre} \in \{0.02, 0.04, 0.06, 0.08, 0.1\}$. The curves for first cycle are similar with those plotted in Verma et al. (2011). In order to obtain the quantitative resemblance of the curves for the subsequent cycles, with those

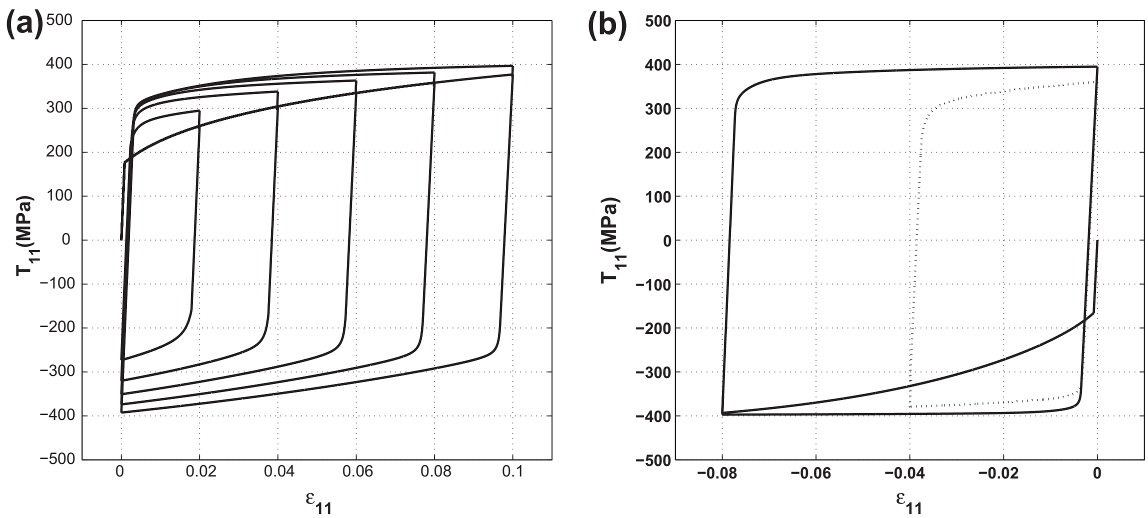

Fig. 3. Strain–stress curves obtained with uniaxial stress model for (a) TCT tests for prestrais at 2%, 4%, 6%, 8%, 10% and (b) CTC tests for prestains at −4% and −8%.

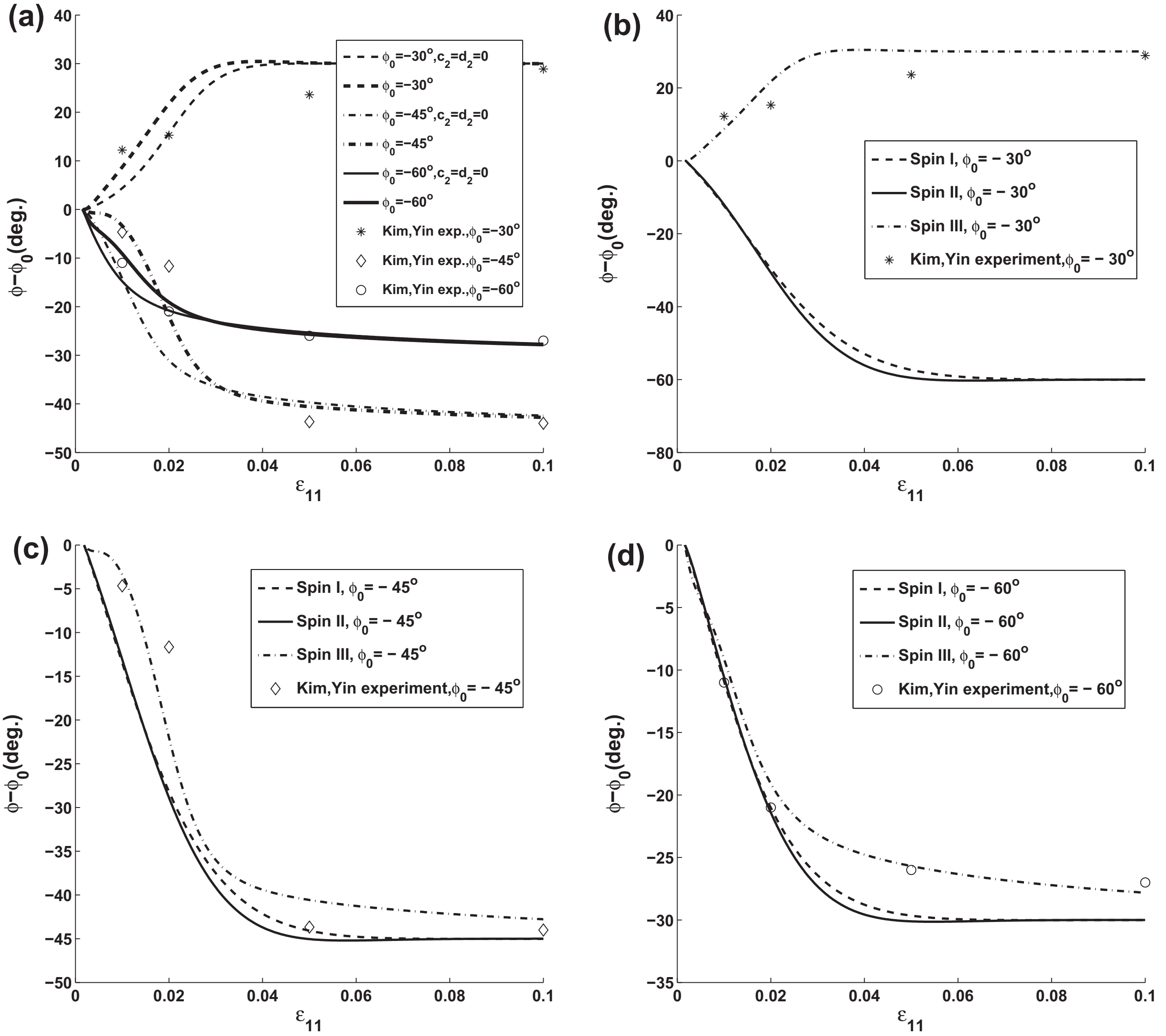

Fig. 4. Variation of Euler's angle $\varphi - \varphi_0$, after the first prestrain at 3% as a function of the axial second strain, ε_{11}, for various plastic spins with the initial conditions (b) $\varphi_0 = -30^\circ$, (c) $\varphi_0 = -45^\circ$ and (d) $\varphi_0 = -60^\circ$. Fig. 4a shows an improvement in the predicted numerical results for $c_2 = 120, d_2 = 250$, when the plastic spin III is considered. The experimentally observed Kim and Yin effect occurs in the plane stress state.

plotted in Figures 8 and 9 from Verma et al. (2011), the dependence of the pre-strains, ε^{pre}, of the kinematic hardening parameter d_1 has been introduced. The kinematic hardening parameter d_1 is introduced under the form $d_1^{soft} = d_1 + (70 - d_1)exp(20(0.1 - \varepsilon^{pre}))$, to emphasize the softening attributed to kinematic hardening.

Plane stress state with the initial orthotropic axis $\mathbf{n}_3 = \mathbf{j}_3$, *namely for* $\theta_0 = 0$*:* We simulate numerically the behaviour of a plate, via the solution of the differential system (59) using Voce-type hardening law, for the deformation process (1), with non-decreasing λ_1 applied along the axis $\mathbf{j}_1$ only. We refer to the plane stress state in order to simulate the experiments used by Kim and Yin (1997). In their experiments, the tensile stretch tests were performed in specimens cut at angles of $30^\circ, 45^\circ, 60^\circ$, respectively, to the rolling directions, from a large sheet pre-strain at 3%. The experimental points determined by Kim and Yin (1997) can be seen in Fig. 4. The monotony of the numerical values of the function φ shows a *rotation in the opposite direction* for $\varphi_0 = -30^\circ$ than that for $\varphi_0 = -45^\circ$ and $\varphi_0 = -60^o$. As we have already mentioned, for the plane stress state and for the prestrain $\varepsilon_{11} = 3\%$ considered in Kim and Yin experiment, the appropriate values of the hardening variables which characterize this prestrain state are calculated and they will be involved in the initial conditions associated with the pre strained state.

Let us remark that in the plane stress state, the appropriate combinations between the spin constants characterize the plastic spins $\hat{\Omega}_{12}$ as it can be seen from the Appendix C. The monotony of the function φ, see (59), is given by the sign of $\hat{\Omega}_{12}$. We look for the spin parameters which lead to $\hat{\Omega}_{12}(t) > 0$ for the initial condition $\varphi_0 = -30^\circ$, and to $\hat{\Omega}_{12}(t) < 0$ for the initial condition $\varphi_0 = -45^\circ$ and $\varphi_0 = -60^\circ$, respectively.

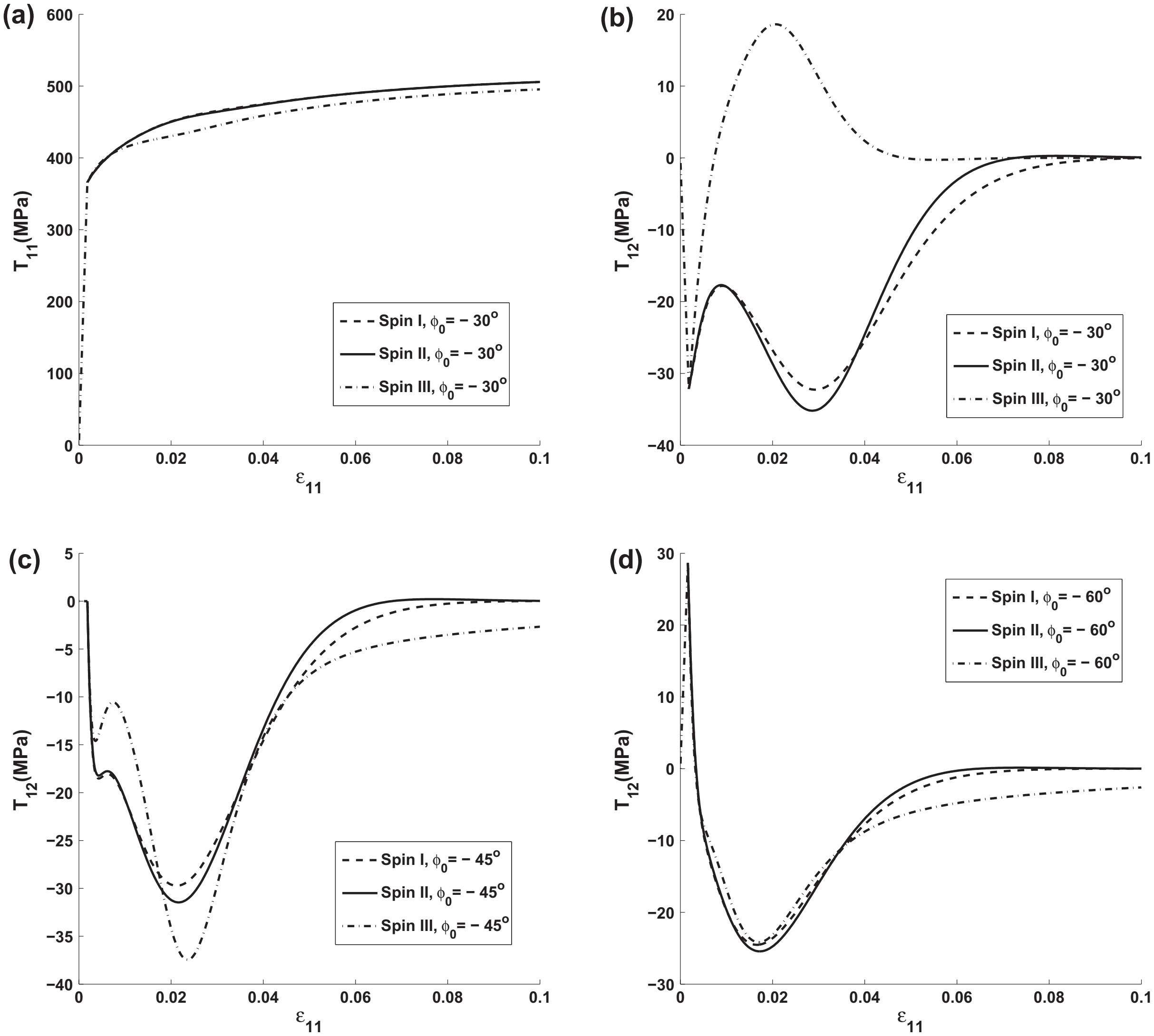

Fig. 5. Influence of the plastic spins (a) on the normal components of the stress T_{11} for $\varphi_0 = -30^\circ$, and on the shear stress component, T_{12}, for the initial conditions (b) $\varphi_0 = -30^\circ$, (c) $\varphi_0 = -45^\circ$, (d) $\varphi_0 = -60^\circ$.

We simulate numerically the *influence of the type of plastic spin* shown in Fig. 4, which present the values of $\varphi - \varphi_0$ as a function of the strain ε_{11}, for the initial conditions $\varphi_0 = -30^\circ$, $\varphi_0 = -45^\circ$ and $\varphi_0 = -60^\circ$, respectively, and all three plastic spins considered herein. There are the set of spin constants which approximate very well the experimental data put into evidence for the spins I and II, for the initial data $\varphi_0 = -45^\circ$ and $\varphi_0 = -60^\circ$, namely $-A_1 + A_2 - A_3 = 0$ and $-A_4 + A_5 - A_6 = -180$ for spin I, $-\eta_1 + \eta_2 - \eta_3 = 0$ and $-\eta_4 + \eta_5 - \eta_6 = -10$ for the spin II. The experimental behaviour observed by Kim and Yin (1997) can be approximated only when the expression of the plastic spin is that generated by $\overline{\mathbf{S}}$ and $\hat{\mathbf{N}}^p$, i.e., the spin III, since there can not be found the constants for the plastic spins I and II which provide an increasing function $\varphi(t)$ for the initial condition $\varphi_0 = -30^\circ$. To obtain the *rotation in the opposite direction* for $\varphi_0 = -30^\circ$ than that for $\varphi_0 = -45^\circ$ and $\varphi_0 = -60^o$ is the ultimate test to choose the type of the spin as Dafalias (2000) noticed. There is the rationale which led to our choice for the plastic spin, namely spin III. We emphasize that the components of the spin III are polynomials of the third order in components of the stress, while the spins I and II are second degree polynomials.

Consequently, in the following, we use the plastic spin III in the numerical simulations of the behaviour of the model presented in this study. The material spin constants are chosen to be in such a way to have $\hat{\eta}_1 = \tilde{\eta} + \tilde{\eta}_2 - \tilde{\eta}_3 = 6502$ and $-\hat{\eta}_2 = \tilde{\eta} + \tilde{\eta}_1 = 7220$ in the formula $(56)_3$.

From Figs. 4 and 5, it can be seen that there is an important influence of the plastic spins on the rotation in the plane angle and on the shear component T_{12}, especially at -30°, in spite of the small amount in its variation. T_{12} tends to stabilize at zero value over $\varepsilon_{11} = 6\%$.

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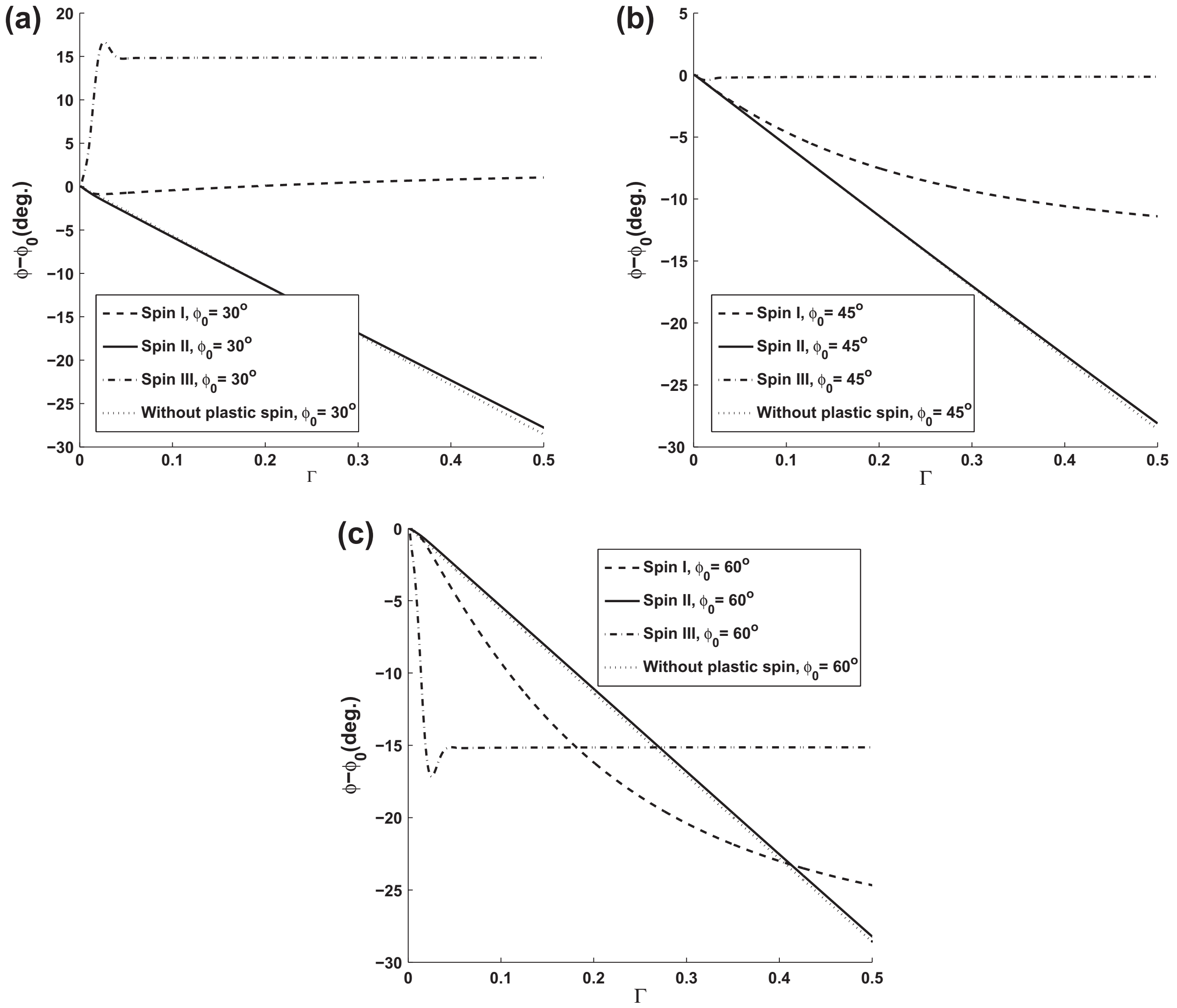

Fig. 6. Variation of Euler's angle as a function of the shear strain Γ plotted for various plastic spins as well as without plastic spin for (a) $\varphi_0 = 30^\circ$, (b) $\varphi_0 = 45^\circ$, (c) $\varphi_0 = 60^\circ$.

Let us remark that in Ulz (2011) the rotation angle of the so called preferred axis is represented versus the uniaxial strain for different values for the plastic spin parameters, but the experimental results by Kim and Yin (1997) appear to be dispersed among the plotted curves, not being localized on these curves that correspond to the appropriate values of plastic spin parameter. On the other hand, when the elastic stretches are large during the process, the orthotropy axes are distorted, namely they do not remain orthogonal, when we pass from the isoclinic configuration to the deformed configuration.

Shear deformation of the plate is numerically simulated via the solution of the differential system (59), when the current value of the homogeneous deformation gradient is given by (81). In this case the *motion spin is not vanishing*. An important influence of plastic spin is emphasized for large amount of shear strains. The stabilization of the orientational anisotropy occurs in the presence of the plastic spin, in contrast with the unreasonable behaviour produced in the absence of the plastic spin. The function $\varphi - \varphi_0$ is plotted in Fig. 6 in terms of the shear deformation Γ for different initial values φ_0, while the non-zero components of the stress are represented in Fig. 7. T_{12} is about ten times higher than T_{11} and T_{22}. The same effect of the non-coaxial plastic spin, i.e., which corresponds to the plastic spin III in an isotropic model, on the shear stress T_{12} and on the normal stress component T_{11}, is emphasized by Kuroda (1996) (for its plastic spin parameter $a = 3$), see the Figs. 1 and 2, in the mentioned paper.

For the *general position of the orthotropic axes*, i.e., when $\sin\theta_0 \neq 0$, the differential system (51) is considered for the homogeneous deformation process (79), with $\lambda_1 \geqslant 0, \lambda_2 = \lambda_3 = 1$, when the plastic spin III and the Voce-type scalar hardening law are involved. The influence of the initial value of φ_0 on the behaviour of the model is analyzed for the initial position of the orthotropy axes described by $\psi_0 = 0^\circ, \theta_0 = 3^\circ$. A significant influence on the values of the Euler angles can be observed,

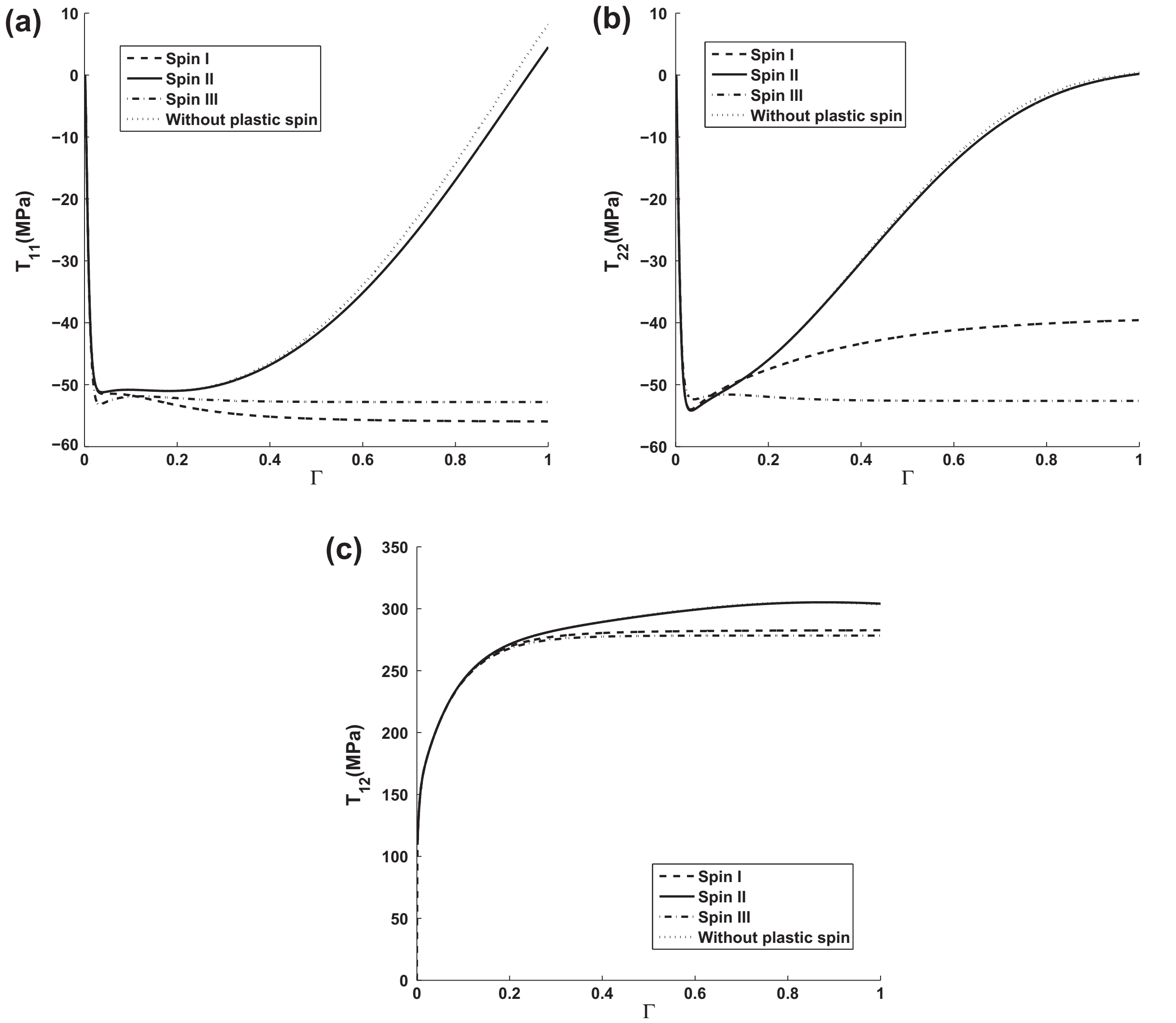

Fig. 7. Variation of (a) T_{11}, (b) T_{22}, and T_{12}, as functions of shear strain Γ, plotted for various plastic spins and as well as without plastic spin for $\varphi_0 = 45^\circ$.

as well as for the stress component T_{13} (similar for T_{23}), see Fig. 8. We remark that their values tend to stabilize at the appropriate values over $\varepsilon_{11} = 4\%$.

Influence of the kinematic hardening on the behaviour of the model could be analyzed based on the numerical solution of the differential system (51) for the general deformation process, when c_0 could be non-zero. We use different values for the constant c_0, say $c_0 = -100$, $c_0 = 100$ and $c_0 = 150$, to observe the influence of the kinematic hardening on the Euler's angle $\theta - \theta_0$, on the stress T_{11} and on the back-stress A_{11}, see Fig. 9, for the initial conditions $\theta_0 = 3^\circ, \psi_0 = 0^\circ, \varphi_0 = 10^\circ$.

8. Conclusions

The model proposed in this paper allows us to describe the variation, with respect to time, of the orthotropic axes, i.e., the so-called orientational anisotropy, starting from the hypothesis that orthotropy is preserved during the deformation process, a fact experimentally motivated by Kim and Yin (1997). If the elastic strains remain small during the deformation process, then the orthotropic axes are rotated only, i.e., they remain orthogonal. The rotation of the orthotropy axes is distinguished from the rotation of the material elements, just the plastic spin makes the difference between the spin of the motion and the spin which characterizes the orientational anisotropy. The change in the anisotropy axes is characterized by the presence of the Euler angles, whose variation in time can be achieved in the absence of the spin of the motion if and only if the plastic spin is involved in the model. When a shear deformation is numerically simulated, the stabilization of the orientational anisotropy occurs in the presence of the plastic spin, in contrast with the unreasonable behaviour produced in the absence of the plastic spin. In this case, the motion spin is non-vanishing.

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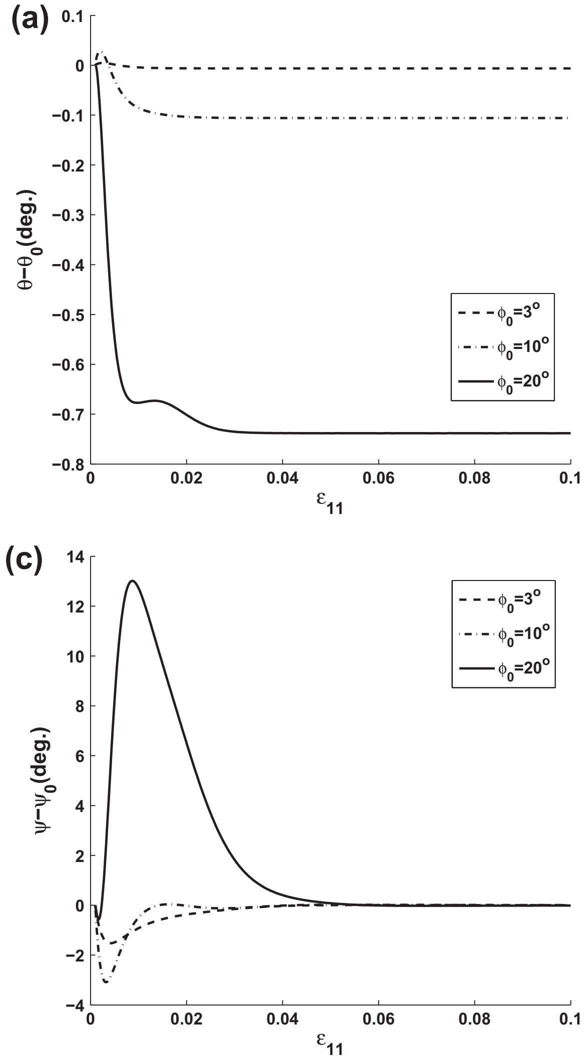

Fig. 8. The variation of Euler's angles (a) $\theta - \theta_0$, (b) $\varphi - \varphi_0$, (c) $\psi - \psi_0$ and stress component (d) T_{13} in terms of ε_{11} with the third spin and for the initial conditions $\psi_0 = 0°, \theta_0 = 3°$, with different φ_0.

The rationale for which the yield function was considered to depend on the third invariant of the effective stress components is motivated by experimental data required to be incorporated in the model, as well as the necessity to have a model which allows for different values of the modulus of the yield stress in compression and traction, the so-called strength differential effect of some metals. We exemplified herein the possibility to describe the three-dimensional behaviour of the model.

The essence of the philosophy concerning the constitutive models makes one confident of the measurements of the parameters in these processes which are allowed within the constitutive framework. As the proposed model is not reduced to in-plane stress or strain processes, parameters such as the elastic constants, yield stresses, parameters of the plastic spin, and so on, are related to the model and not to the processes.

The flexibility of the proposed model with the third invariant of the stress in the yield function allows us to model the strength differential effect in the initial yield condition on one hand and determine the yield coefficients which are compatible with the set of experimental anisotropic parameters traditionally reported in the literature on the other hand. The material coefficients describing the initial yield functions and hardening variables were determined based on experimental data in a plane stress process and in the uniaxial cyclic tests reported by Verma et al. (2011). We included also tension, compression and reversal test data to emphasize the strength-differential and kinematic hardening effects.

Once the elastic, yield and hardening material constants have been determined, the analysis of the behaviour of the solution to the differential system, which describes the evolution in time of the stress, hardening variable and motion of

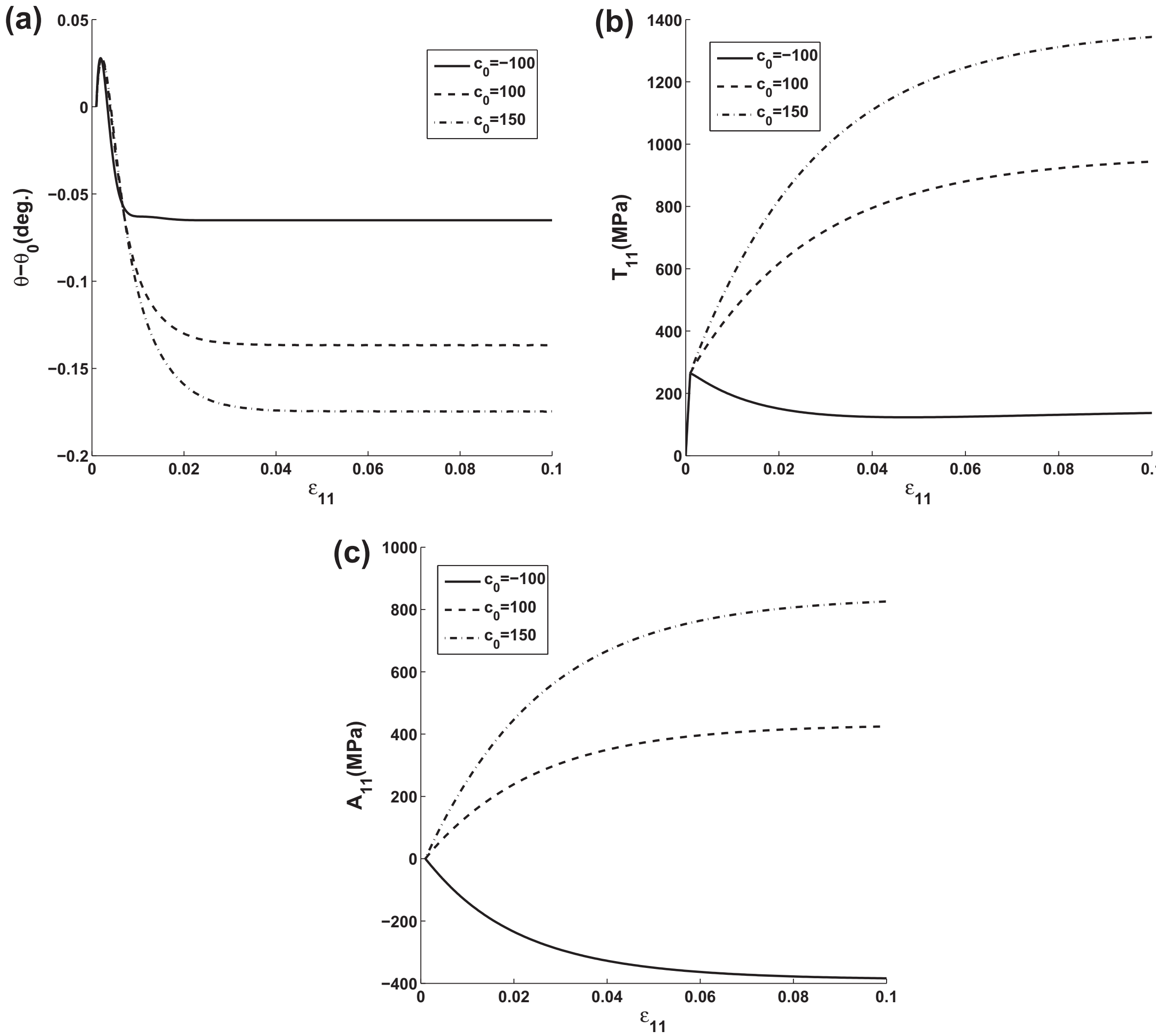

Fig. 9. The variation of Euler's angle (a) $\theta - \theta_0$, (b) stress component T_{11} and (c) hardening component A_{11} in terms of ε_{11} with the third spin, for the initial conditions $\psi_0 = 0^\circ$, $\theta_0 = 3^\circ$, $\varphi_0 = 10^\circ$ and for different values c_0.

the anisotropy axes, makes possible to completely define the model, having a special selection of the plastic spin and hardening variables. Only when the expression of the plastic spin is that generated by $\bar{\mathbf{S}}$ and $\hat{\mathbf{N}}^p$, i.e., the spin III, it is possible to obtain a rotation in the opposite direction for $\varphi_0 = -30^\circ$ unlike the case when considering $\varphi_0 = -45^\circ$ and $\varphi_0 = -60^\circ$. We note that, in the present modelling, the same set of plastic spin constants have been chosen for the three graphs unlike the papers by Dafalias (2000), Ulz (2011) and Han et al. (2002), where the best curves are fitted for different constants values describing the spins. In the mentioned papers the spin description corresponds to isotropic expression derived from our formula (37), for spin III, with only one non-zero parameter $\tilde{\eta}$. Han et al. (2002) used an additional parameter which enters the expression of $\tilde{\eta} \equiv \mu^{\Phi}$, which is dependent on the initial position of the appropriate axis.

The behaviour of this model is strongly influenced by various material parameters, the initial condition, or the process, e.g., a plane stress state or a plane strain state.

The behaviour of the material under the plane stress and plane strain, as well as in unidimensional cyclic homogeneous processes, have also been analyzed therein.

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The very constructive comments and suggestions made by the anonymous reviewers are gratefully acknowledged.

Appendix A

$$
\begin{aligned}
&K_{11} = C_1 + C_2 + C_4 + C_7 + C_8, \quad K_{22} = C_1 + C_3 + C_5 + C_7 + C_9,\ K_{33} = C_1 + C_7,\\
&K_{m1} = 2C_1 + C_2 + C_3, \quad K_{m2} = 2C_1 + C_2, \quad K_{m3} = 2C_1 + C_3,\\
&K_{12} = C_6 + 2C_7 + C_8 + C_9, \quad K_{13} = 2C_7 + C_8, \quad K_{23} = 2C_7 + C_9.
\end{aligned} \tag{A1}
$$

The coefficients $k_i, i = 1, \ldots, 20$, are functions of the yield material constants $B_j, j = 1, \ldots, 10$:

$$
\begin{aligned}
k_1 &= \frac{1}{9}(3B_1 - 3B_2 + B_3 - 2B_4 - 3B_7 + 9B_8 + 3B_9)\\
k_2 &= \frac{1}{9}(-3B_1 + 3B_2 + 3B_4 + B_5 + B_6 + 3B_7 + 9B_8 + 3B_{10}),\\
k_3 &= \frac{1}{3}(-B_1 - B_2 + 3B_8), \quad k_4 = \frac{1}{18}(-9B_1 + 15B_2 - 3B_3 + 15B_4 + B_5 + B_6 + 18B_7 - 27B_8 - 9B_9 + 3B_{10})\\
k_5 &= \frac{1}{18}(-9B_1 + 3B_2 - 3B_3 - 3B_4 - B_5 - B_6 - 27B_8 - 9B_9 - 3B_{10})\\
k_6 &= \frac{1}{18}(15B_1 - 9B_2 + B_3 - 17B_4 - 3B_5 - 3B_6 - 18B_7 - 27B_8 + 3B_9 - 9B_{10})\\
k_7 &= \frac{1}{18}(3B_1 - 9B_2 - B_3 - B_4 - 3B_5 - 3B_6 - 27B_8 - 3B_9 - 9B_{10})\\
k_8 &= \frac{1}{18}(15B_1 + 3B_2 + B_3 + B_4 - B_5 - B_6 - 27B_8 + 3B_9 - 3B_{10})\\
k_9 &= \frac{1}{18}(3B_1 + 15B_2 - B_3 - B_4 + B_5 + B_6 - 27B_8 - 3B_9 + 3B_{10})\\
k_{10} &= \frac{1}{18}(15B_1 - 21B_2 + 13B_3 - 5B_4 + 13B_5 - 5B_6 + 81B_8 + 21B_9 + 3B_{10})
\end{aligned} \tag{A2}
$$

$$
\begin{aligned}
k_{11} &= \frac{1}{18}(-21B_1 + 15B_2 - 5B_3 + 13B_4 - 5B_5 + 13B_6 + 81B_8 + 3B_9 + 21B_{10})\\
k_{12} &= \frac{1}{9}(3B_1 + 3B_2 - 4B_3 - 4B_4 - 4B_5 - 4B_6 - 81B_8 - 12B_9 - 12B_{10})\\
k_{13} &= \frac{1}{18}(15B_1 - 21B_2 + 13B_3 - 5B_4 + B_5 + B_6 + 81B_8 + 21B_9 + 3B_{10})\\
k_{14} &= \frac{1}{9}(3B_1 + 21B_2 - 4B_3 + 5B_4 - B_5 - B_6 - 81B_8 - 12B_9 - 3B_{10})\\
k_{15} &= \frac{1}{18}(-21B_1 - 21B_2 - 5B_3 - 5B_4 + B_5 + B_6 + 81B_8 + 3B_9 + 3B_{10})\\
k_{16} &= \frac{1}{9}(21B_1 + 3B_2 - B_3 - B_4 + 5B_5 - 4B_6 - 81B_8 - 3B_9 - 12B_{10})\\
k_{17} &= \frac{1}{18}(-21B_1 + 15B_2 + B_3 + B_4 - 5B_5 + 13B_6 + 81B_8 + 3B_9 + 21B_{10})\\
k_{18} &= \frac{1}{18}(-21B_1 - 21B_2 + B_3 + B_4 - 5B_5 - 5B_6 + 81B_8 + 3B_9 + 3B_{10})\\
k_{19} &= \frac{1}{9}(6B_1 - 6B_2 + 2B_3 + 2B_4 + 2B_5 + 2B_6 + 54B_8 + 6B_9 + 6B_{10})\\
k_{20} &= \frac{1}{3}(-9B_1 - 9B_2 + B_3 + B_4 + B_5 + B_6 + 81B_8 + 9B_9 + 9B_{10})
\end{aligned} \tag{A3}
$$

Remark. Only six material constants are needed for the plain stress state case, namely $k_1, k_2, k_4, k_6, k_{10}$, and k_{11}.
The linear orthotropic elastic type constitutive equation written for $\hat{\mathcal{E}}[\mathbf{D}]$ in the orthotropic basis is given by

$$
\begin{aligned}
&(\hat{\mathcal{E}}[\mathbf{D}])_{11} = a_{11}D_{11} + a_{12}D_{22} + a_{13}D_{33}, \quad (\hat{\mathcal{E}}[\mathbf{D}])_{12} = a_{44}D_{12},\\
&(\hat{\mathcal{E}}[\mathbf{D}])_{22} = a_{12}D_{11} + a_{22}D_{22} + a_{23}D_{33}, \quad (\hat{\mathcal{E}}[\mathbf{D}])_{13} = a_{66}D_{13},\\
&(\hat{\mathcal{E}}[\mathbf{D}])_{33} = a_{13}D_{11} + a_{23}D_{22} + a_{33}D_{33}, \quad (\hat{\mathcal{E}}[\mathbf{D}])_{23} = a_{55}D_{23}.
\end{aligned} \tag{A4}
$$

For $a_{11} = a_{22} = a_{33},\ a_{12} = a_{13} = a_{23},\ a_{44} = a_{55} = a_{66}$, one obtains the special case of materials with cubic symmetry, see Ting (1996). The relationships between the elastic constants are given by

$$
a_{11} = \frac{E(1-\nu)}{1-\nu-2\nu^2}, \quad a_{12} = \frac{E\nu}{1-\nu-2\nu^2}, \quad a_{44} = \mu,
$$

where E is Young's modulus and ν Poisson's ratio.

The components of $\hat{\mathbf{N}}^p(\frac{\mathbf{T}}{\rho}, \mathbf{A}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)$ with respect to the basis $(\mathbf{m}_i \otimes \mathbf{m}_j)$ are given by

$$\begin{aligned}
(\widehat{N}^p)_{11} &= \frac{3}{2}\sqrt{f_2}(2K_{11}\bar{S}_{11} + (K_{33} - K_{11} - K_{22})\bar{S}_{22} + (K_{22} - K_{11} - K_{33})\bar{S}_{33}) \\
&\quad - \gamma(3k_1\bar{S}_{11}^2 + 2k_4\bar{S}_{11}\bar{S}_{22} + 2k_5\bar{S}_{11}\bar{S}_{33} + k_6\bar{S}_{22}^2 + k_8\bar{S}_{33}^2 + k_{10}\bar{S}_{12}^2 + \\
&\quad + k_{13}\bar{S}_{13}^2 + k_{16}\bar{S}_{23}^2 + k_{19}\bar{S}_{22}\bar{S}_{33}) \\
(\widehat{N}^p)_{22} &= \frac{3}{2}\sqrt{f_2}((K_{33} - K_{11} - K_{22})\bar{S}_{11} + 2K_{22}\bar{S}_{22} + (K_{11} - K_{22} - K_{33})\bar{S}_{33}) \\
&\quad - \gamma(3k_2\bar{S}_{22}^2 + k_4\bar{S}_{11}^2 + 2k_6\bar{S}_{11}\bar{S}_{22} + 2k_7\bar{S}_{22}\bar{S}_{33} + k_9\bar{S}_{33}^2 + k_{11}\bar{S}_{12}^2 + \\
&\quad + k_{14}\bar{S}_{13}^2 + k_{17}\bar{S}_{23}^2 + k_{19}\bar{S}_{11}\bar{S}_{33}) \\
(\widehat{N}^p)_{33} &= \frac{3}{2}\sqrt{f_2}((K_{22} - K_{11} - K_{33})\bar{S}_{11} + (K_{11} - K_{22} - K_{33})\bar{S}_{22} + 2K_{33}\bar{S}_{33}) \\
&\quad - \gamma(3k_3\bar{S}_{33}^2 + k_5\bar{S}_{11}^2 + k_7\bar{S}_{22}^2 + 2k_8\bar{S}_{11}\bar{S}_{33} + 2k_9\bar{S}_{22}\bar{S}_{33} + \\
&\quad + k_{12}\bar{S}_{12}^2 + k_{15}\bar{S}_{13}^2 + k_{18}\bar{S}_{23}^2 + k_{19}\bar{S}_{11}\bar{S}_{22}) \\
(\widehat{N}^p)_{12} &= \frac{3}{2}\sqrt{f_2}K_{m1}\bar{S}_{12} - \gamma(k_{10}\bar{S}_{11}\bar{S}_{12} + k_{11}\bar{S}_{22}\bar{S}_{12} + k_{12}\bar{S}_{33}\bar{S}_{12} + \frac{1}{2}k_{20}\bar{S}_{13}\bar{S}_{23}) \\
(\widehat{N}^p)_{13} &= \frac{3}{2}\sqrt{f_2}K_{m2}\bar{S}_{13} - \gamma(k_{13}\bar{S}_{11}\bar{S}_{13} + k_{14}\bar{S}_{22}\bar{S}_{13} + k_{15}\bar{S}_{33}\bar{S}_{13} + \frac{1}{2}k_{20}\bar{S}_{12}\bar{S}_{23}) \\
(\widehat{N}^p)_{23} &= \frac{3}{2}\sqrt{f_2}K_{m3}\bar{S}_{23} - \gamma(k_{16}\bar{S}_{11}\bar{S}_{23} + k_{17}\bar{S}_{22}\bar{S}_{23} + k_{18}\bar{S}_{33}\bar{S}_{23} + \frac{1}{2}k_{20}\bar{S}_{12}\bar{S}_{13})
\end{aligned} \tag{A5}$$

The expression of the plastic multiplier, β, is obtained using $\widehat{N}^p_{ij}$ and $\widetilde{D}_{ij}$ as follows

$$\begin{aligned}
\beta &= (a_{11}\widehat{N}^p_{11} + a_{12}\widehat{N}^p_{22} + a_{13}\widehat{N}^p_{33})\widetilde{D}_{11} + (a_{12}\widehat{N}^p_{11} + a_{22}\widehat{N}^p_{22} + a_{23}\widehat{N}^p_{33})\widetilde{D}_{22} + (a_{13}\widehat{N}^p_{11} + a_{23}\widehat{N}^p_{22} + a_{33}\widehat{N}^p_{33})\widetilde{D}_{33} \\
&\quad + 2a_{44}\widehat{N}^p_{12}\widetilde{D}_{12} + 2a_{66}\widehat{N}^p_{13}\widetilde{D}_{13} + 2a_{55}\widehat{N}^p_{23}\widetilde{D}_{23}
\end{aligned} \tag{A6}$$

Note that function h_c is expressed using $\widehat{N}^p_{ij}$, $\hat{l}_{ij}$ and $\hat{b}$, namely

$$\begin{aligned}
h_c &= \widehat{N}^p_{11}(a_{11}\widehat{N}^p_{11} + a_{12}\widehat{N}^p_{22} + a_{13}\widehat{N}^p_{33} + \hat{l}_{11}) \\
&\quad + \widehat{N}^p_{22}(a_{12}\widehat{N}^p_{11} + a_{22}\widehat{N}^p_{22} + a_{23}\widehat{N}^p_{33} + \hat{l}_{22}) + \widehat{N}^p_{33}(a_{13}\widehat{N}^p_{11} + a_{23}\widehat{N}^p_{22} + a_{33}\widehat{N}^p_{33} + \hat{l}_{33}) \\
&\quad + 2\widehat{N}^p_{12}(a_{44}\widehat{N}^p_{12} + \hat{l}_{12}) + 2\widehat{N}^p_{13}(a_{66}\widehat{N}^p_{13} + \hat{l}_{13}) + 2\widehat{N}^p_{23}(a_{55}\widehat{N}^p_{23} + \hat{l}_{23}) + \partial_\kappa F(\kappa)\hat{b}
\end{aligned} \tag{A7}$$

Remark. For plane stress case we use in this expressions the restrictions $S_{13} = S_{23} = S_{33} = 0$ and for uniaxial stress case $S_{22} = S_{12} = S_{13} = S_{23} = S_{33} = 0$.

Appendix B

The stretching $\mathbf{D}$ has the components $\widetilde{D}_{ij}$ with respect to the actual orthotropic axes, expressed in terms of the components of the rotation tensor $\mathbf{R}$ and the components $\mathbf{D}$ written in the basis $\mathbf{j}_i \otimes \mathbf{j}_k$, $\widetilde{D}_{ij} = (\mathbf{R}^T\mathbf{D}\mathbf{R})_{ij}$, given by

$$\begin{aligned}
\widetilde{D}_{11} &= R_{11}^2D_{11} + R_{21}^2D_{22} + R_{31}^2D_{33} + 2R_{21}R_{11}D_{12} + 2R_{11}R_{31}D_{13} + 2R_{31}R_{21}D_{23} \\
\widetilde{D}_{22} &= R_{12}^2D_{11} + R_{22}^2D_{22} + R_{32}^2D_{33} + 2R_{12}R_{22}D_{12} + 2R_{12}R_{32}D_{13} + 2R_{22}R_{32}D_{23} \\
\widetilde{D}_{33} &= R_{13}^2D_{11} + R_{23}^2D_{22} + R_{33}^2D_{33} + 2R_{13}R_{23}D_{12} + 2R_{13}R_{33}D_{13} + 2R_{23}R_{33}D_{23} \\
\widetilde{D}_{12} &= R_{11}R_{12}D_{11} + R_{21}R_{22}D_{22} + R_{31}R_{32}D_{33} + (R_{21}R_{12} + R_{11}R_{22})D_{12} \\
&\quad + (R_{31}R_{12} + R_{11}R_{32})D_{13} + (R_{31}R_{22} + R_{21}R_{32})D_{23} \\
\widetilde{D}_{13} &= R_{11}R_{13}D_{11} + R_{21}R_{23}D_{22} + R_{31}R_{33}D_{33} + (R_{11}R_{23} + R_{21}R_{13})D_{12} \\
&\quad + (R_{11}R_{33} + R_{31}R_{13})D_{13} + (R_{31}R_{23} + R_{21}R_{33})D_{23} \\
\widetilde{D}_{23} &= R_{12}R_{13}D_{11} + R_{22}R_{23}D_{22} + R_{32}R_{33}D_{33} + (R_{12}R_{23} + R_{13}R_{22})D_{12} \\
&\quad + (R_{12}R_{33} + R_{13}R_{32})D_{13} + (R_{22}R_{33} + R_{23}R_{32})D_{23}
\end{aligned} \tag{B1}$$

Remark. For a plane rotation of the orthotropy axes $R_{13} = R_{23} = R_{31} = R_{32} = 0$.

In the case of the evolution equation for the tensorial hardening variable of the Armstrong-Frederick type adapted to the orthoropic symmetry, the following representation for the components $\hat{l}_{ij} := \mathbf{m}_i \cdot \hat{\mathbf{l}}\mathbf{m}_j$, in terms of $\widehat{N}^p_{ij}$ and A_{ij}, is obtained

$$\begin{aligned}
&\hat{l}_{11} = (c_0 + 2c_1)\widehat{N}^p_{11} - \hat{b}(d_0 + 2d_1)A_{11}, \quad \hat{l}_{22} = (c_0 + 2c_2)\widehat{N}^p_{22} - \hat{b}(d_0 + 2d_2)A_{22}, \\
&\hat{l}_{33} = c_0\widehat{N}^p_{33} - \hat{b}d_0A_{33},\ \hat{l}_{12} = (c_0 + c_1 + c_2)\widehat{N}^p_{12} - \hat{b}(d_0 + d_1 + d_2)A_{12}, \\
&\hat{l}_{13} = (c_0 + c_1)\widehat{N}^p_{13} - \hat{b}(d_0 + d_1)A_{13},\ \hat{l}_{23} = (c_0 + c_2)\widehat{N}^p_{23} - \hat{b}(d_0 + d_2)A_{23}.
\end{aligned} \tag{B2}$$

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Proof of the Theorem 6. We consider the differential system (51). We start from the initial conditions that correspond to an elastic state, which means that the stress process remain inside the initial elastic domain a certain time interval, say $[t_0, t_1)$, i.e., $\hat{\mu} = 0$, $\mathbf{a} = 0, \kappa = 0, \mathbf{R}^e = \mathbf{R}(t_0)$. $\mathbf{R}(t_0)$ is dependent on the initial values of Euler's angles, $\theta(t_0) = 0, \psi(t_0) = 0$ and $\varphi(t_0) = \varphi_0$. We denote by t_1 the moment of time at which the stress state reaches the initial yield surface. Since the shear components of the stretching, $D_{13} = D_{23}$, vanish, then $\widetilde{D}_{13}(t) = 0$ and $\widetilde{D}_{23}(t) = 0$ on the same time interval as a consequence of the formulae given in (B1). If $\hat{\mu}(t_0) > 0$, from the differential Eqs. (51), which refer to the shear stress components T_{13} and T_{23}, and to $A_{13} = A_{23}$, together with $\theta = 0$ and with the appropriate components for $\widehat{N}^p_{ij}$ given in (A5), and with (B1), we obtain that the unique solution has the vanishing components $T_{13} = T_{23} = 0$, and $A_{13} = A_{23} = 0$. Consequently, $\overline{S}_{13} = \overline{S}_{23} = 0$, while $\hat{\mu}$, β and h_c can be found from (A6), (A7). Next, we focus on Euler's angles. First we remark that $\hat{\Omega}^p_{13} = \hat{\Omega}^p_{23} = 0$ as a consequence of expressions for $\hat{\mathbf{\Omega}}^p_{ij}$ given in (C1)-(C3). Thus from the differential system (51), together with the initial condition $\theta(t_0) = 0$, it follows that $\theta(t) = 0$ satisfies the appropriate differential equation. Consequently, the anisotropy axes could support only a plane rotation. ψ is considered to be zero and only one of Euler's angles, namely φ, characterizes the plane rotation.

Appendix C

Herein we present the non-vanishing components of the plastic spin for all the types of spin considered in the paper. The plastic spin generated by $\mathbf{S}$, with respect to the basis $(\mathbf{m}_i \otimes \mathbf{m}_j)$, is given by

$$\begin{aligned}
\hat{\Omega}^p_{12} &= (-A_1 + A_2 - A_3)\overline{S}_{12} + (-A_4 + A_5 - A_6)(\overline{S}_{11}\overline{S}_{12} + \overline{S}_{12}\overline{S}_{22} + \overline{S}_{13}\overline{S}_{23}) \\
\hat{\Omega}^p_{13} &= -A_1\overline{S}_{13} - A_4(\overline{S}_{11}\overline{S}_{13} + \overline{S}_{12}\overline{S}_{23} + \overline{S}_{13}\overline{S}_{33}) \\
\hat{\Omega}^p_{23} &= -A_2\overline{S}_{23} - A_5(\overline{S}_{12}\overline{S}_{13} + \overline{S}_{22}\overline{S}_{23} + \overline{S}_{23}\overline{S}_{33})
\end{aligned} \tag{C1}$$

The components of the plastic spin generated by $\mathbf{N}^p$, with respect to the axes $(\mathbf{m}_i \otimes \mathbf{m}_j)$, are given by

$$\begin{aligned}
\hat{\Omega}^p_{12} &= (-\eta_1 + \eta_2 - \eta_3)\widehat{N}^p_{12} + (-\eta_4 + \eta_5 - \eta_6)(\widehat{N}^p_{11}\widehat{N}^p_{12} + \widehat{N}^p_{12}\widehat{N}^p_{22} + \widehat{N}^p_{13}\widehat{N}^p_{23}) \\
\hat{\Omega}^p_{13} &= -\eta_1\widehat{N}^p_{13} - \eta_4(\widehat{N}^p_{11}\widehat{N}^p_{13} + \widehat{N}^p_{12}\widehat{N}^p_{23} + \widehat{N}^p_{13}\widehat{N}^p_{33}) \\
\hat{\Omega}^p_{23} &= -\eta_2\widehat{N}^p_{23} - \eta_5(\widehat{N}^p_{12}\widehat{N}^p_{13} + \widehat{N}^p_{22}\widehat{N}^p_{23} + \widehat{N}^p_{23}\widehat{N}^p_{33})
\end{aligned} \tag{C2}$$

When considering the plastic spin generated by $\mathbf{S}$ and $\mathbf{N^p}$, then its components, again with respect to the actual orthotropic axes $(\mathbf{m}_i \otimes \mathbf{m}_j)$, are given by

$$\begin{aligned}
\hat{\Omega}^p_{12} &= (\tilde{\eta} + \tilde{\eta}_2 - \tilde{\eta}_3)(\overline{S}_{11}\widehat{N}^p_{12} + \overline{S}_{12}\widehat{N}^p_{22} + \overline{S}_{13}\widehat{N}^p_{23}) - (\tilde{\eta} + \tilde{\eta}_1)(\overline{S}_{12}\widehat{N}^p_{11} + \overline{S}_{22}\widehat{N}^p_{12} + \overline{S}_{23}\widehat{N}^p_{13}) \\
\hat{\Omega}^p_{13} &= \tilde{\eta}(\overline{S}_{11}\widehat{N}^p_{13} + \overline{S}_{12}\widehat{N}^p_{23} + \overline{S}_{13}\widehat{N}^p_{33}) + (-\tilde{\eta} - \tilde{\eta}_1)(\overline{S}_{13}\widehat{N}^p_{11} + \overline{S}_{23}\widehat{N}^p_{12} + \overline{S}_{33}\widehat{N}^p_{13}) \\
\hat{\Omega}^p_{23} &= \tilde{\eta}(\overline{S}_{12}\widehat{N}^p_{13} + \overline{S}_{22}\widehat{N}^p_{23} + \overline{S}_{23}\widehat{N}^p_{33}) + (-\tilde{\eta} - \tilde{\eta}_2)(\overline{S}_{13}\widehat{N}^p_{12} + \overline{S}_{23}\widehat{N}^p_{22} + \overline{S}_{33}\widehat{N}^p_{23})
\end{aligned} \tag{C3}$$

Appendix D

E_1 and E_2 which have been introduced in formula (72) are calculated in terms of α and ξ, and explicitly given by

$$\begin{aligned}
E_1 &= \frac{3}{2}\sqrt{f_2|_\alpha}\,[K_{33} - K_{11} - K_{22} + 2(2K_{11} + 2K_{22} - K_{33} - K_{m1})\cos^2\alpha\sin^2\alpha] - \\
&\quad - \gamma[k_4(\xi)\cos^6\alpha + (3k_1(\xi) + 2k_6(\xi) - 2k_{10}(\xi) + k_{11}(\xi))\cos^4\alpha\sin^2\alpha + \\
&\quad + (3k_2(\xi) + 2k_4(\xi) + k_{10}(\xi) - 2k_{11}(\xi))\cos^2\alpha\sin^4\alpha + k_6(\xi)\sin^6\alpha]; \\
E_2 &= \frac{3}{2}\sqrt{f_2|_\alpha}\,[2K_{11}\cos^2\alpha + 2K_{22}\sin^2\alpha + K_{33} - K_{11} - K_{22}] - \gamma[(3k_1(\xi) + k_4(\xi))\cos^4\alpha + \\
&\quad + (3k_2(\xi) + k_6(\xi))\sin^4\alpha + (2k_4(\xi) + 2k_6(\xi) + k_{10}(\xi) + k_{11}(\xi))\cos^2\alpha\sin^2\alpha]
\end{aligned} \tag{D1}$$

where $f_2|_\alpha$ is given by (71).

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