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Two-level methods for variational and quasi-variational inequalities of the second kind

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Two-level methods for variational and quasi-variational inequalities of the second kind

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Abstract

We introduce and analyze some two-level multiplicative and additive Schwarz methods for variational and quasi-variational inequalities of the second kind. The methods are introduced as subspace correction algorithms for problems in a reflexive Banach space. We prove that these methods are globally convergent and give, under some assumptions, error estimates. In the finite element spaces, the introduced algorithms are in fact two-level Schwarz methods. In this case we prove that the assumptions we made for the general convergence result hold, and write the convergence rate depending on the overlapping and mesh parameters. We get that our methods have an optimal convergence rate, it is almost independent of the mesh and overlapping parameters, and also, the methods have an optimal computing complexity per iteration.

Keywords: domain decomposition methods, multilevel methods, subspace correction methods, variational and quasi-variational inequalities of the second kind.
AMS subject classification: 65N55, 65K15, 65N30.

1 Introduction

Literature on the Schwarz methods is very large, and it is motivated by their capability in providing robust and efficient algorithms for large scale problems. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting in 1987 with [12] or those cited in the books [16], [19], [20] and [21]. Naturally, most of the papers dealing with these methods are dedicated to the linear problems. However, their generalization to non–linear problems is

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not straightforward, in particular for variational inequalities of the second kind or for quasi-variational inequalities, is far from being trivial. The convergence of the projected Gauss–Seidel relaxation (or successive coordinate minimization) for variational inequalities of the second kind in $\mathbf{R}^d$ has been proved in [11]. There, the non-differentiable term has been decomposed as a sum of terms, each of them depending only on one vector component. The projected Gauss-Seidel method is a particular case of a Schwarz method in which the domain is decomposed into the interior of the supports of the nodal basis functions. Consequently, the above representation of the non-differentiable term can be viewed as a decomposition in concordance with the domain decomposition. A straightforward generalization of the convergence proof in [11] to more general decompositions can be obtained using this idea, but it fails if, in order to get a faster convergence, a two-level or multilevel method is considered. This is due to the fact that the nonlinearities are not decoupled on the coarser levels. A remedy can be found in adapting minimization techniques for the construction and analysis of multigrid and domain decomposition methods, see [14]–[16].

In [7] (see also [5] and [6]), one- and two-level multiplicative Schwarz methods have been proposed for variational and quasi-variational inequalities of the second kind, and they have been applied to frictional contact problems. It is proved there that the convergence rates of the two-level methods are almost independent of the mesh and overlapping parameters. However, the original convex set, which is defined on the fine grid, is used to find the corrections on the coarse grid, too. This leads to a suboptimal computing complexity. To avoid visiting the fine grid, some approximating subsets of this convex set for the coarse levels have been constructed in [10], [18], [10] and [14]–[16] for complementarity problems. It is well-known that the additive methods are the best on parallel machines even if their convergence is a little slower than that of the multiplicative ones. In this paper, we introduce multiplicative and additive two-level methods for variational and quasi-variational inequalities of the second kind whose convex set is of two-obstacle type. Suitable constraints for the corrections computed on the coarse mesh are provided in order to ensure the optimal convergence of the methods. In this way, besides the optimal convergence rate, these methods have also an optimal computing complexity.

The paper is organized as follows. Section 2 is devoted to a general framework in a reflexive Banach space. We introduce here an assumption on the construction of the level convex sets. Another two hypotheses will be introduced, which will be necessary in the convergence proofs, one for the multiplicative algorithms and the other one for the additive ones. Mainly, these hypotheses refer to the decomposition of the elements in the convex set, and introduce a constant C_0 which will play an important role in the writing of the convergence rate. In Section 3, we introduce subspace correction algorithms for variational inequalities of the second kind, and prove that, under the above assumptions, they are globally convergent. We also estimate their convergence rates. In Section 4, we introduce subspace correction algorithms for the quasi-variational inequalities. As in the previous section, we prove their convergence and estimate the convergence rate, using the assumptions introduced in Section 2. Section 5 is devoted to the two-level methods. If we associate finite element subspaces to the domain decomposition and to the coarse grid, the abstract algorithms introduced in Sections 3 and 4 become two-level Schwarz methods. We show that the assumptions introduced in the previous sections hold for two-obstacle convex sets and we explicitly write the constant C_0 depending on the mesh and domain decomposition parameters. In this way, we get that the convergence rates of the two-level methods for the variational and quasi-variational inequalities of the second kind are similar with the convergence rates obtained for equations, ie., we get an optimal convergence. In the case of the two-level methods, the convergence rate is almost independent of the mesh and domain decomposition parameters.

2 General framework

Let V be a reflexive Banach space and $V_0, V_{11}, \cdots, V_{1m}$ be some closed subspaces of V. Subspace V_0 will correspond to the coarse discretization, and $V_{11}, \cdots, V_{1m}$ corresponds to the decomposition of the domain. Also, let $K \subset V$ be a non empty closed convex set of V. To introduce the algorithms, we make an assumption on choice of the convex sets where we look for the level corrections. These level convex sets depend on the current approximation in the algorithms.

Assumption 2.1. *We assume that for a given $w \in K$, we can recursively introduce the convex sets K_1 and K_0 as:*
$0 \in K_1,\ K_1 \subset \{v_1 \in V : w + v_1 \in K\}$ *and, for a* $w_1 \in K_1$,
$0 \in K_0,\ K_0 \subset \{v_0 \in V_0 : w + w_1 + v_0 \in K\}$.

As we already said, we shall analyze both types of algorithms, multiplicative and additive. In the case of the multiplicative algorithms we make the following

Assumption 2.2. *There exists a constant $C_0 > 0$ such that for any $u, w \in K$, any $w_{1i} \in V_{1i}$, $w_{11} + \ldots + w_{1i} \in K_1$, $i = 1, \ldots, m$, and any $w_0 \in K_0$, there exist $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$, which satisfy*

$$u_{11} \in K_1 \text{ and } w_{11} + \ldots + w_{1i-1} + u_{1i} \in K_1,\ i = 2, \ldots, m,\ u_0 \in K_0$$
$$u - w = \sum_{i=1}^{m} u_{1i} + u_0 \text{ and}$$
$$\sum_{i=1}^{m} ||u_{1i}|| \le C_0(||u - w|| + \sum_{i=1}^{m} ||w_{1i}|| + ||w_0||).$$

The convex sets K_1 and K_0 are constructed as in Assumption 2.1 using w and $w_1 = w_{11} + \ldots + w_{1m}$.

This assumption is simpler in the case of the additive algorithms

ASSUMPTION 2.3. *There exist a constant $C_0 > 0$ such that for any $u, w \in K$, there exist $u_{1i} \in V_{1i} \cap K_1$, $i = 1, \ldots, m$, and $u_0 \in K_0$, which satisfy*

$$u - w = \sum_{i=1}^{m} u_{1i} + u_0 \text{ and } \sum_{i=1}^{m} ||u_{1i}|| + ||u_0|| \leq C_0 ||u - w||.$$

The convex sets K_1 and K_0 are constructed as in Assumption 2.1 with the above w and $w_1 = 0$.

Now, we consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$, and assume that there exist two real numbers $p,\ q > 1$ such that for any real number $M > 0$ there exist two constants $\alpha_M,\ \beta_M > 0$ for which

$$\alpha_M \|v - u\|^p \leq \langle F'(v) - F'(u), v - u \rangle, \text{ and} \tag{2.1}$$

$$\|F'(v) - F'(u)\|_{V'} \leq \beta_M \|v - u\|^{q-1}, \tag{2.2}$$

for any $u, v \in V$ with $\|u\|, \|v\| \leq M$. Above, we have denoted by F' the Gâteaux derivative of F, and we have marked that the constants α_M and β_M may depend on M. It is evident that if (2.1) and (2.2) hold, then for any $u, v \in V$, $\|u\|, \|v\| \leq M$, we have

$$\alpha_M \|v - u\|^p \leq \langle F'(v) - F'(u), v - u \rangle \leq \beta_M \|v - u\|^q. \tag{2.3}$$

Following the way in [13], we can prove that for any $u, v \in V$, $\|u\|, \|v\| \leq M$, we have

$$\begin{array}{l} \langle F'(u), v - u \rangle + \frac{\alpha_M}{p} \|v - u\|^p \leq F(v) - F(u) \leq \\ \langle F'(u), v - u \rangle + \frac{\beta_M}{q} \|v - u\|^q. \end{array} \tag{2.4}$$

We point out that since F is Gâteaux differentiable and satisfies (2.4), F is a strictly convex functional (see Proposition 5.4 in [9], pag. 24). Also, we can prove that $q \leq 2 \leq p$.

3 Subspace correction algorithm for variational inequalities of the second kind

Let $\varphi : V \to \mathbf{R}$ be a convex lower semicontinuous functional and we assume that $F + \varphi$ is coercive in the sense that

$$F(v) + \varphi(v) \to \infty, \text{ as } \|v\| \to \infty, \ v \in K, \tag{3.1}$$

if K is not bounded. In the multiplicative case, in addition to the hypotheses of Assumption 2.2, we suppose that

$$\begin{aligned} &\textstyle\sum_{i=1}^{m}[\varphi(w+\sum_{j=1}^{i-1} w_{1j}+u_{1i})-\varphi(w+\sum_{j=1}^{i-1} w_{1j}+w_{1i})]+ \\ &\varphi(w+w_1+u_0)-\varphi(w+w_1+w_0)\le \\ &\varphi(u)-\varphi(w+\textstyle\sum_{i=1}^{m} w_{1i}+w_0) \end{aligned} \tag{3.2}$$

for $u,w\in K$, $u_{1i},w_{1i}\in V_{1i}$ and $u_0,w_0\in V_0$ as in Assumption 2.2. Also, in addition to Assumption 2.3, for the additive case, we suppose that

$$\sum_{i=1}^{m}\varphi(w+u_{1i})+\varphi(w+u_0)\le m\varphi(w)+\varphi(u) \tag{3.3}$$

for any $u,w\in K$, $u_{1i}\in V_{1i}$, $i=1,\dots,m$, and $u_0\in V_0$ which satisfy Assumption 2.3.

Now, we consider the problem

$$u\in K:\ \langle F'(u),v-u\rangle+\varphi(v)-\varphi(u)\ge 0,\ \text{for any } v\in K, \tag{3.4}$$

which is equivalent with the minimization problem

$$u\in K:\ F(u)+\varphi(u)\le F(v)+\varphi(v),\ \text{for any } v\in K. \tag{3.5}$$

These problems have a unique solution (see [9], Proposition 1.2, pag. 34). From (2.4) we see that, for a given $M>0$ such that the solution u of (3.4) satisfies $\|u\|\le M$, we have

$$\begin{aligned} &\tfrac{\alpha_M}{p}\|v-u\|^p\le F(v)-F(u)+\varphi(v)-\varphi(u), \\ &\text{for any } v\in K,\ \|v\|\le M. \end{aligned} \tag{3.6}$$

We first introduce the algorithm which is of the multiplicative type

ALGORITHM 3.1. *We start the algorithm with an arbitrary* $u^0\in K$. *Assuming that at iteration* $n\ge 0$ *we have* $u^n\in K$, *we successively perform the following steps:*

- at the level 1, as in Assumption 2.1, with $w=u^n$, *we construct the convex set* K_1. *Then, we first write* $w_1^n=0$, *and, for* $i=1,\dots,m$, *we successively calculate* $w_{1i}^{n+1}\in V_{1i}$, $w_1^{n+\frac{i-1}{m}}+w_{1i}^{n+1}\in K_1$, *the solution of the inequalities*

$$\begin{aligned} &\langle F'(u^n+w_1^{n+\frac{i-1}{m}}+w_{1i}^{n+1}),v_{1i}-w_{1i}^{n+1}\rangle+ \\ &\varphi(u^n+w_1^{n+\frac{i-1}{m}}+v_{1i})-\varphi(u^n+w_1^{n+\frac{i-1}{m}}+w_{1i}^{n+1})\ge 0, \end{aligned} \tag{3.7}$$

for any $v_{1i}\in V_{1i}$, $w_1^{n+\frac{i-1}{m}}+v_{1i}\in K_1$, *and write* $w_1^{n+\frac{i}{m}}=w_1^{n+\frac{i-1}{m}}+w_{1i}^{n+1}$,

- at the level 0*, we construct, as in Assumption 2.1 with* $w = u^n$ *and* $w_1 = w_1^{n+1}$*, the convex set* K_0*. Then, we calculate* $w_0^{n+1} \in K_0$*, the solution of the inequality*

$$(3.8)\qquad \begin{aligned}&\langle F'(u^n + w_1^{n+1} + w_0^{n+1}), v_0 - w_0^{n+1}\rangle + \\ &\varphi(u^n + w_1^{n+1} + v_0) - \varphi(u^n + w_1^{n+1} + w_0^{n+1}) \geq 0,\end{aligned}$$

for any $v_0 \in K_0$*,*
- we write $u^{n+1} = u^n + w_1^{n+1} + w_0^{n+1}$*.*

The proposed additive algorithm is written as follows

Algorithm 3.2. *We start the algorithm with an* $u^0 \in K$*. Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$*, we simultaneously perform, the following steps:*
- we construct the convex sets K_1 *and* K_0 *as in Assumption 2.1 with* $w = u^n$ *and* $w_1 = 0$*,*
- we simultaneously calculate,
$\cdot$ $w_{1i}^{n+1} \in V_{1i} \cap K_1$*, the solutions of the inequalities*

$$(3.9)\quad \langle F'(u^n + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1}\rangle + \varphi(u^n + v_{1i}) - \varphi(u^n + w_{1i}^{n+1}) \geq 0,$$

for any $v_{1i} \in V_{1i} \cap K_1$*, write* $w_1^{n+1} = \sum_{i=1}^m w_{1i}^{n+1}$*, and*
$\cdot$ $w_0^{n+1} K_0$*, the solution of the inequality*

$$(3.10)\quad \langle F'(u^n + w_0^{n+1}), v_0 - w_0^{n+1}\rangle + \varphi(u^n + v_0) - \varphi(u^n + w_0^{n+1}) \geq 0,$$

for any $v_0 \in K_0$*,*
Then, we write $u^{n+1} = u^n + \frac{r}{m+1}(w_1^{n+1} + w_0^{n+1})$*, with a fixed* $0 < r \leq 1$*.*

These algorithms do not suppose a decomposition of the convex set K depending on the subspaces of V. Like problem (3.4), problems (3.7)–(3.10) have unique solutions, and they are equivalent with minimization problems. We have the following general convergence result.

Theorem 3.1. *Let* V *be a reflexive Banach,* $V_0, V_{11}, \cdots, V_{1m}$ *some closed subspaces of* V*, and* K *a non empty closed convex subset of* V *which satisfies Assumption 2.1, Assumption 2.2 when we apply Algorithm 3.1, and Assumption 2.3 in the case of Algorithm 3.2. Also, we assume that* F *is Gâteaux differentiable and satisfies (2.1) and (2.2), the functional* φ *is convex and lower semicontinuous, satisfies (3.2) for Algorithm 3.1, (3.3) for Algorithm 3.2, and* $F + \varphi$ *is coercive if* K *is not bounded. Let*

$$(3.11)\qquad M = \sup\{||v|| : F(v) + \varphi(v) \leq F(u^0) + \varphi(u^0)\}$$

where u^0 *is the starting point in Algorithms 3.1 or 3.2. Then, the norms of the approximations of the solution* u *of problem (3.4) obtained from* *these algorithms are bounded by M and we have the following error estimations:*

(i) if $p = q = 2$ we have

$$\begin{aligned} &F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \\ &(\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)], \end{aligned} \tag{3.12}$$

$$\|u^n - u\|^2 \leq \tfrac{2}{\alpha_M}(\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)]. \tag{3.13}$$

(ii) if $p > q$ we have

$$\begin{aligned} &F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \\ &\tfrac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}, \end{aligned} \tag{3.14}$$

$$\|u - u^n\|^p \leq \tfrac{p}{\alpha_M} \tfrac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}. \tag{3.15}$$

The constants $C_1 > 0$ and $C_2 > 0$ depend on the functional F, the solution u, the initial approximation u^0, m, and the constant C_0.

Remark 3.1. *For Algorithm 3.1, constants C_1 and C_2 can be written as,*

$$\begin{aligned} C_1 = \beta_M(1+2C_0)(m+1)^{2-\frac{q}{p}}(\tfrac{p}{\alpha_M})^{\frac{q}{p}}(F(u^0) - F(u) \\ +\varphi(u^0) - \varphi(u))^{\frac{p-q}{p(p-1)}} + \beta_M C_0 (m+1)^{\frac{p-q+1}{p}} \tfrac{1}{\varepsilon^{\frac{1}{p-1}}}(\tfrac{p}{\alpha_M})^{\frac{q-1}{p-1}} \end{aligned} \tag{3.16}$$

$$C_2 = \frac{p-q}{(p-1)(F(u^0) + \varphi(u^0) - F(u) - \varphi(u))^{\frac{p-q}{q-1}} + (q-1)C_1^{\frac{p-1}{q-1}}} \tag{3.17}$$

where

$$\varepsilon = \alpha_M / (p\beta_M C_0 (m+1)^{\frac{p-q+1}{p}}) . \tag{3.18}$$

Also, in the case of Algorithm 3.2, these constants can be written as,

$$\begin{aligned} &C_1 = \tfrac{m+1}{r}[1 - \tfrac{r}{m+1} + (1+C_0)(m+1)\tfrac{\beta_M}{\frac{\alpha_M}{2}} + \\ &C_0^2(m+1)(\tfrac{\beta_M}{\frac{\alpha_M}{2}})^2] \end{aligned} \tag{3.19}$$

$$\begin{aligned} &C_2 = \\ &\tfrac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}+(q-1)C_3^{\frac{p-1}{q-1}}}. \end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
&C_3 = \tfrac{m+1-r}{r}[F(u^0) - F(u) + \varphi(u^0) - \varphi(u)]^{\frac{p-q}{p-1}} + \\
&(\tfrac{m+1}{r})^{\frac{q}{p}} \tfrac{\beta_M(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \cdot \\
&(F(u^0) - F(u) + \varphi(u^0) - \varphi(u))^{\frac{p-q}{p(p-1)}} + \\
&(\tfrac{m+1}{r})^{\frac{q-1}{p-1}} \tfrac{\beta_M^{\frac{p}{p-1}} C_0^{\frac{p}{p-1}} (m+1)^{q-1}}{(\frac{\alpha_M}{p})^{\frac{q}{p-1}}}
\end{aligned} \tag{3.21}$$

Proof of Theorem 3.1. Except the changes of notation due to the introduction of the convex sets K_1 and K_0, the proof in the case of the multiplicative Algorithm 3.1 is identical with that of Theorem 1 in [7] (see also [5]) and will be omitted. Also, the proof for the additive Algorithm 3.2 uses the same techniques as that given for the minimization of non-quadratic functionals in [4]. The proof is divided into several steps.

Step 1. The existence of M defined in (3.11) follows from the coercivity of $F + \varphi$. In view of the convexity of F, we get

$$\begin{aligned}
&F(u^{n+1}) = F(u^n + \tfrac{r}{m+1}(\textstyle\sum_{i=1}^m w_{1i}^{n+1} + w_0^{n+1})) = \\
&F((1-r)u^n + \tfrac{r}{m+1}(\textstyle\sum_{i=1}^m (u^n + w_{1i}^{n+1}) + u^n + w_0^{n+1})) \leq \\
&(1-r)F(u^n) + \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m F(u^n + w_{1i}^{n+1}) + F(u^n + w_0^{n+1})]
\end{aligned}$$

A similar result can be obtained for φ, ie., we have

$$\begin{aligned}
&F(u^{n+1}) \leq (1-r)F(u^n) + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m F(u^n + w_{1i}^{n+1}) + F(u^n + w_0^{n+1})] \\
&\varphi(u^{n+1}) \leq (1-r)\varphi(u^n) + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m \varphi(u^n + w_{1i}^{n+1}) + \varphi(u^n + w_0^{n+1})]
\end{aligned} \tag{3.22}$$

From (3.9), (3.10) and these inequalities, we get

$$F(u^{n+1}) + \varphi(u^{n+1}) \leq F(u^n) + \varphi(u^n)$$

Therefore, for any $n \geq 0$ and $i = 1, \cdots, m$, we get

$$\begin{aligned}
&\max\{F(u^n + w_{1i}^{n+1}) + \varphi(u^n + w_{1i}^{n+1}), \\
&F(u^n + w_0^{n+1}) + \varphi(u^n + w_0^{n+1})\} \leq \\
&F(u^n) + \varphi(u^n) \leq F(u^0) + \varphi(u^0).
\end{aligned} \tag{3.23}$$

Step 2. Now, from (3.9), (3.10) and (2.4), for any $n \geq 0$ and $i = 1, \cdots, m$, we have

$$\begin{aligned}
&F(u^n) - F(u^n + w_{1i}^{n+1}) + \varphi(u^n) - \varphi(u^n + w_{1i}^{n+1}) \geq \\
&\tfrac{\alpha_M}{p}\|w_{1i}^{n+1}\|^p \text{ and} \\
&F(u^n) - F(u^n + w_0^{n+1}) + \varphi(u^n) - \varphi(u^n + w_0^{n+1}) \geq \\
&\tfrac{\alpha_M}{p}\|w_0^{n+1}\|^p
\end{aligned} \tag{3.24}$$

In view of (3.22) and (3.24), we get

$$\begin{aligned}&F(u^{n+1}) \leq (1-r)F(u^n) + \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m F(u^n + w_{1i}^{n+1}) + F(u^n + w_0^{n+1})] \leq \\ &F(u^n) - \tfrac{r}{m+1}\tfrac{\alpha_M}{p}[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p] + \\ &\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u^n) - \varphi(u^n + w_{1i}^{n+1})) + \varphi(u^n) - \varphi(u^n + w_0^{n+1})]\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\tfrac{r}{m+1}\tfrac{\alpha_M}{p}[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p] \leq F(u^n) - F(u^{n+1}) \\ &\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u^n) - \varphi(u^n + w_{1i}^{n+1})) + \varphi(u^n) - \varphi(u^n + w_0^{n+1})]\end{aligned} \tag{3.25}$$

But, in view of (3.22), we have

$$\begin{aligned}&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u^n) - \varphi(u^n + w_{1i}^{n+1})) + \varphi(u^n) - \varphi(u^n + w_0^{n+1})] \leq \\ &\varphi(u^n) - \varphi(u^{n+1}),\end{aligned}$$

and consequently,

$$\begin{aligned}&\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p \leq \\ &\tfrac{m+1}{r}\tfrac{p}{\alpha_M}[F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})]\end{aligned} \tag{3.26}$$

Step 3. Writing

$$\tilde{u}^{n+1} = u^n + \sum_{i=1}^m w_{1i}^{n+1} + w_0^{n+1}, \tag{3.27}$$

from the convexity of F, we get

$$F(u^{n+1}) \leq (1 - \frac{r}{m+1})F(u^n) + \frac{r}{m+1}F(\tilde{u}^{n+1}) \tag{3.28}$$

Applying Assumption 2.3 for $w = u^n$ and $v = u$, we get a decomposition $u_{11}^n, \cdots, u_{1m}^n$, u_0^n, of $u - u^n$, and we can replace v_{1i} and v_0 by u_{1i}^n and u_0^n in (3.9) and (3.10), respectively. From (3.28), (2.4), (3.9) and (3.10), we obtain

$$\begin{aligned}&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \tfrac{r}{m+1}\tfrac{\alpha_M}{p}||u - \tilde{u}^{n+1}||^p \leq \\ &(1 - \tfrac{r}{m+1})[F(u^n) - F(u)] + \\ &\tfrac{r}{m+1}[F(\tilde{u}^{n+1}) - F(u) + \tfrac{\alpha_M}{p}||u - \tilde{u}^{n+1}||^p] + \varphi(u^{n+1}) - \varphi(u) \leq \\ &(1 - \tfrac{r}{m+1})[F(u^n) - F(u)] + \\ &\tfrac{r}{m+1}\langle F'(\tilde{u}^{n+1}), \tilde{u}^{n+1} - u\rangle + \varphi(u^{n+1}) - \varphi(u) \leq \\ &(1 - \tfrac{r}{m+1})[F(u^n) - F(u)] + \\ &\tfrac{r}{m+1}\textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle + \\ &\tfrac{r}{m+1}\langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle + \\ &\tfrac{r}{m+1}\textstyle\sum_{i=1}^m [\varphi(u^n + u_{1i}^n) - \varphi(u^n + w_{1i}^{n+1})] + \\ &\tfrac{r}{m+1}[\varphi(u^n + u_0^n) - \varphi(u^n + w_0^{n+1})] + \varphi(u^{n+1}) - \varphi(u)\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&F(u^{n+1})-F(u)+\varphi(u^{n+1})-\varphi(u)+\tfrac{r}{m+1}\tfrac{\alpha_M}{p}||u-\tilde{u}^{n+1}||^p \le \\
&(1-\tfrac{r}{m+1})[F(u^n)-F(u)+\varphi(u^n)-\varphi(u)]+ \\
&\tfrac{r}{m+1}\textstyle\sum_{i=1}^m\langle F'(u^n+w_{1i}^{n+1})-F'(\tilde{u}^{n+1}),u_{1i}^n-w_{1i}^{n+1}\rangle+ \\
&\tfrac{r}{m+1}\langle F'(u^n+w_0^{n+1})-F'(\tilde{u}^{n+1}),u_0^n-w_0^{n+1}\rangle+ \\
&\tfrac{r}{m+1}\textstyle\sum_{i=1}^m[\varphi(u^n+u_{1i}^n)-\varphi(u^n+w_{1i}^{n+1})]+ \\
&\tfrac{r}{m+1}[\varphi(u^n+u_0^n)-\varphi(u^n+w_0^{n+1})] \\
&\tfrac{r}{m+1}[\varphi(u^n)-\varphi(u)]+\varphi(u^{n+1})-\varphi(u^n)
\end{aligned}\tag{3.29}$$

As in [4], using (2.2) and Assumption 2.3, we get

$$\begin{aligned}
&\textstyle\sum_{i=1}^m\langle F'(u^n+w_{1i}^{n+1})-F'(\tilde{u}^{n+1}),u_{1i}^n-w_{1i}^{n+1}\rangle+ \\
&\langle F'(u^n+w_0^{n+1})-F'(\tilde{u}^{n+1}),u_0^n-w_0^{n+1}\rangle \le \\
&\beta_M(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||+||w_0^{n+1}||)^{q-1}[\textstyle\sum_{i=1}^m||u_{1i}^n-w_{1i}^{n+1}||+||u_0^n-w_0^{n+1}||] \le \\
&\beta_M(m+1)^{\frac{(p-1)(q-1)}{p}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q-1}{p}}\cdot \\
&[\textstyle\sum_{i=1}^m(||u_{1i}^n||+||w_{1i}^{n+1}||)+||u_0^n||+||w_0^{n+1}||] \le \\
&\beta_M(m+1)^{\frac{(p-1)(q-1)}{p}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q-1}{p}}\cdot \\
&(C_0||u-u^n||+\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||+||w_0^{n+1}||) \le \\
&\beta_M(m+1)^{\frac{(p-1)(q-1)}{p}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q-1}{p}}\cdot \\
&(C_0||u-\bar{u}^{n+1}||+(1+C_0)(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||+||w_0^{n+1}||)) \le \\
&\beta_M C_0(m+1)^{\frac{(p-1)(q-1)}{p}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q-1}{p}}||u-\bar{u}^{n+1}||+ \\
&\beta_M(1+C_0)(m+1)^{\frac{(p-1)q}{p}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q}{p}}
\end{aligned}$$

But, for any $\varepsilon>0$, $r>1$ and $x,y\ge 0$, we have $x^{\frac{1}{r}}y\le\varepsilon x+\frac{1}{\varepsilon^{\frac{1}{r-1}}}y^{\frac{r}{r-1}}$. Therefore, we get

$$\begin{aligned}
&\textstyle\sum_{i=1}^m\langle F'(u^n+w_{1i}^{n+1})-F'(\tilde{u}^{n+1}),u_{1i}^n-w_{1i}^{n+1}\rangle+ \\
&\langle F'(u^n+w_0^{n+1})-F'(\tilde{u}^{n+1}),u_0^n-w_0^{n+1}\rangle \le \\
&\beta_M(1+C_0)(m+1)^{\frac{(p-1)q}{p}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q}{p}}+ \\
&\beta_M C_0\frac{(m+1)^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}}(\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^p+||w_0^{n+1}||^p)^{\frac{q-1}{p-1}}+ \\
&\beta_M C_0\varepsilon(m+1)^{\frac{(p-1)(q-1)}{p}}||u-\bar{u}^{n+1}||^p
\end{aligned}\tag{3.30}$$

for any $\varepsilon>0$. Also, using (3.22) and (3.3), we get

$$\begin{aligned}
&\tfrac{r}{m+1}\textstyle\sum_{i=1}^m[\varphi(u^n+u_{1i}^n)-\varphi(u^n+w_{1i}^{n+1})]+ \\
&\tfrac{r}{m+1}[\varphi(u^n+u_0^n)-\varphi(u^n+w_0^{n+1})]+ \\
&\tfrac{r}{m+1}[\varphi(u^n)-\varphi(u)]+\varphi(u^{n+1})-\varphi(u^n) \le \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m\varphi(u^n+u_{1i}^n)+\varphi(u^n+u_0^n)-m\varphi(u^n)-\varphi(u)]\le 0
\end{aligned}$$

From (3.29) and (3.30), we have

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \\
&\tfrac{r}{m+1}[\tfrac{\alpha_M}{p} - \beta_M C_0 \varepsilon (m+1)^{\frac{(p-1)(q-1)}{p}}] ||u - \tilde{u}^{n+1}||^p \leq \\
&(1 - \tfrac{r}{m+1})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] + \\
&\tfrac{r}{m+1}\beta_M [(1 + C_0)(m+1)^{\frac{(p-1)q}{p}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q}{p}} + \\
&C_0 \frac{(m+1)^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q-1}{p-1}}]
\end{aligned}
\tag{3.31}
$$

for any $\varepsilon > 0$.

Step 4. From (3.31) and (3.26), we get

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \\
&\tfrac{r}{m+1}[\tfrac{\alpha_M}{p} - \beta_M C_0 \varepsilon (m+1)^{\frac{(p-1)(q-1)}{p}}] ||u - \tilde{u}^{n+1}||^p \leq \\
&(1 - \tfrac{r}{m+1})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] + \\
&\tfrac{r}{m+1}\beta_M [(\tfrac{m+1}{r})^{\frac{q}{p}} \tfrac{(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \cdot \\
&(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q}{p}} + \\
&(\tfrac{m+1}{r})^{\frac{q-1}{p-1}} \tfrac{C_0 (m+1)^{\frac{(p-1)(q-1)}{p}}}{(\frac{\alpha_M}{p})^{\frac{q-1}{p-1}} \varepsilon^{\frac{1}{p-1}}} \cdot \\
&(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q-1}{p-1}}]
\end{aligned}
$$

With

$$
\varepsilon = \frac{\alpha_M}{p} \frac{1}{\beta_M C_0 (m+1)^{\frac{(p-1)(q-1)}{p}}},
$$

the above equation becomes,

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) \leq \\
&\tfrac{m+1-r}{r}[F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})] + \\
&\beta_M [(\tfrac{m+1}{r})^{\frac{q}{p}} \tfrac{(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \cdot \\
&(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q}{p}} + \\
&(\tfrac{m+1}{r})^{\frac{q-1}{p-1}} \tfrac{\beta_M^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}} (m+1)^{q-1}}{(\frac{\alpha_M}{p})^{\frac{q}{p-1}}} \cdot \\
&(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q-1}{p-1}}]
\end{aligned}
\tag{3.32}
$$

Using (3.6), we see that error estimations in (3.13) and (3.15) can be obtained from (3.12) and (3.14), respectively.

Now, if $p = q = 2$, from the above equation, we easily get equation (3.12), where C_1 is given in (3.19).

Finally, if $q < p$, from (3.22), (3.23) and (3.32), we get

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) \leq \\ &C_3[F(u^n) + \varphi(u^n) - F(u^{n+1}) - \varphi(u^{n+1})]^{\frac{q-1}{p-1}}. \end{aligned} \tag{3.33}$$

where C_3 is given in (3.21). Now, from (3.33), we get

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) + \frac{1}{C_3^{\frac{p-1}{q-1}}}[F(u^{n+1}) + \varphi(u^{n+1}) - \\ &F(u) - \varphi(u)]^{\frac{p-1}{q-1}} \leq F(u^n) + \varphi(u^n) - F(u) - \varphi(u), \end{aligned}$$

and, like in [2] or [4], we have

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) \leq \\ &[(n+1)C_2 + (F(u^0) + \varphi(u^0) - F(u) - \varphi(u))^{\frac{q-p}{q-1}}]^{\frac{q-1}{q-p}}, \end{aligned} \tag{3.34}$$

where C_2 is given in (3.20). Equation (3.34) is another form of (3.14). $\square$

4 Subspace correction algorithms for quasi-variational inequalities

Let $\varphi : V \times V \to \mathbf{R}$ be a functional such that, for any $u \in K$, $\varphi(u, \cdot) : K \to \mathbf{R}$ is convex and lower semicontinuous. We assume that $F + \varphi$ is coercive in the sense that

$$F(v) + \varphi(u, v) \to \infty, \text{ as } \|v\| \to \infty, \ v \in K, \text{ for any } u \in K \tag{4.1}$$

if K is not bounded.

In this section we assume that $p = q = 2$ in (2.1) and (2.2). Also, we assume that for any $M > 0$ there exists $c_M > 0$ such that

$$\begin{aligned} &|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \leq \\ &c_M \|v_1 - v_2\| \|w_1 - w_2\| \end{aligned} \tag{4.2}$$

for any $v_1, \ v_2, \ w_1 \ w_2 \in K$, $\|v_1\|, \ \|v_2\|, \ \|w_1\| \ \|w_2\| \leq M$. As in the previous section, we introduce additional conditions concerning φ. In the multiplicative case, we suppose that

$$\begin{aligned} &\textstyle\sum_{i=1}^{m}[\varphi(u, w + \sum_{j=1}^{i-1} w_{1j} + u_{1i}) - \varphi(u, w + \sum_{j=1}^{i-1} w_{1j} + w_{1i})] + \\ &\varphi(u, w + w_1 + u_0) - \varphi(w + w_1 + w_0) \leq \\ &\varphi(u, v) - \varphi(u, w + \textstyle\sum_{i=1}^{m} w_{1i} + w_0) \end{aligned} \tag{4.3}$$

for $u, w \in K$, $u_{1i}, w_{1i} \in V_{1i}$ and $u_0, w_0 \in V_0$ satisfying Assumption 2.2. Also, for the additive case, we suppose that

$$\sum_{i=1}^{m} \varphi(u, w + u_{1i}) + \varphi(u, w + u_0) \leq m\varphi(u, w) + \varphi(u, u) \tag{4.4}$$

for any $u, w \in K$, $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$ which satisfy Assumption 2.3.

Now, we consider the quasi-variational inequality

$$u \in K : \langle F'(u), v - u\rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \text{ for any } v \in K. \tag{4.5}$$

Since φ is convex in the second variable and F is differentiable and satisfies (2.1), problem (4.5) is equivalent with the minimization problem

$$u \in K : F(u) + \varphi(u, u) \leq F(v) + \varphi(u, v), \text{ for any } v \in K. \tag{4.6}$$

As in [7], we can show that problem (4.5) has a unique solution if there exists a constant $\varkappa < 1$ such that $\frac{c_M}{\alpha_M} \leq \varkappa$, for any $M > 0$. In view of (2.4) we see that, for a given $M > 0$ such that the solution u of (4.5) satisfies $\|u\| \leq M$, we have

$$\begin{aligned} &\tfrac{\alpha_M}{2}\|v - u\|^2 \leq F(v) - F(u) + \varphi(u, v) - \varphi(u, u), \\ &\text{for any } v \in K, \ \|v\| \leq M. \end{aligned} \tag{4.7}$$

To solve problem (4.5), we can introduce three multiplicative algorithms. The first one can be written as,

Algorithm 4.1. *We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we successively perform the following steps:*

- at the level 1, as in Assumption 2.1, with $w = u^n$, we construct the convex set K_1. Then, we first write $w_1^n = 0$, and, for $i = 1, \ldots, m$, we successively calculate $w_{1i}^{n+1} \in V_{1i}$, $w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1} \in K_1$, the solution of the inequalities

$$\begin{aligned} &\langle F'(u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1}\rangle + \\ &\varphi(v_{1i}^{n+1}, u^n + w_1^{n+\frac{i-1}{m}} + v_{1i}) - \\ &\varphi(v_{1i}^{n+1}, u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}) \geq 0, \end{aligned} \tag{4.8}$$

for any $v_{1i} \in V_{1i}$, $w_1^{n+\frac{i-1}{m}} + v_{1i} \in K_1$, and write $w_1^{n+\frac{i}{m}} = w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}$. Above, the first argument of φ is

$$v_{1i}^{n+1} = u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}. \tag{4.9}$$

- at the level 0, as in Assumption 2.1, we construct the convex set K_0 with $w = u^n$ and $w_1 = w_1^{n+1}$. Then, we calculate $w_0^{n+1} \in K_0$, the solution of the inequality

$$\begin{aligned} &\langle F'(u^n + w_1^{n+1} + w_0^{n+1}), v_0 - w_0^{n+1}\rangle + \\ &\varphi(v_0^{n+1}, u^n + w_1^{n+1} + v_0) - \varphi(v_0^{n+1}, u^n + w_1^{n+1} + w_0^{n+1}) \geq 0, \end{aligned} \tag{4.10}$$

for any $v_0 \in K_0$, *where*

$$v_0^{n+1} = u^n + w_1^{n+1} + w_0^{n+1}. \tag{4.11}$$

- we write $u^{n+1} = u^n + w_1^{n+1} + w_0^{n+1}$.

The other algorithms are variants of the above algorithm in which we change the first argument of φ, taking

$$v_{1i}^{n+1} = u^n + w_1^{n+\frac{i-1}{m}} \text{ and } v_0^{n+1} = u^n + w_1^{n+1} \tag{4.12}$$

or

$$v_{1i}^{n+1} = v_0^{n+1} = u^n \tag{4.13}$$

Also, we introduce two additive algorithms. A first algorithm corresponding to the subspaces V_0, $V_{11}, \cdots, V_1m$ and the convex set K is written as follows

Algorithm 4.2. *We start the algorithm with an* $u^0 \in K$. *Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$, *we simultaneously perform, the following steps:*

- we construct the convex sets K_1 *and* K_0 *as in Assumption 2.1 with* $w = u^n$ *and* $w_1 = 0$,

- we simultaneously calculate,

$\cdot$ $w_{1i}^{n+1} \in V_{1i} \cap K_1$, *the solutions of the inequalities*

$$\begin{aligned}&\langle F'(u^n + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1}\rangle + \\ &\varphi(v_{1i}^{n+1}, u^n + v_{1i}) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1}) \geq 0,\end{aligned} \tag{4.14}$$

for any $v_{1i} \in V_{1i} \cap K_1$, *write* $w_1^{n+1} = \sum_{i=1}^m w_{1i}^{n+1}$, *and*

$\cdot$ $w_0^{n+1} K_0$, *the solution of the inequality*

$$\begin{aligned}&\langle F'(u^n + w_0^{n+1}), v_0 - w_0^{n+1}\rangle + \\ &\varphi(v_0^{n+1}, u^n + v_0) - \varphi(v_0^{n+1}, u^n + w_0^{n+1}) \geq 0,\end{aligned} \tag{4.15}$$

for any $v_0 \in K_0$, *where*

$$v_{1i}^{n+1} = v_0^{n+1} = u^n + w_{1i}^{n+1}. \tag{4.16}$$

Then, we write $u^{n+1} = u^n + \frac{r}{m+1}(w_1^{n+1} + w_0^{n+1})$, *with a fixed* $0 < r \leq 1$.

A simplified variant of Algorithm 4.2 is obtained by taking

$$v_{1i}^{n+1} = u^n + w_{1i}^{n+1} \text{ and } v_0^{n+1} = u^n + w_0^{n+1}. \tag{4.17}$$

Like for problem (4.5), we can prove that the problems in the above algorithms are equivalent with minimization problems, and they have unique solutions if there exists a constant $\varkappa < 1$ such that $\frac{c_M}{\alpha_M} \leq \varkappa$, for any $M > 0$.

The following theorem proves that if c_M is small enough, then Algorithms 4.1, 4.2 and their variants are convergent.

Theorem 4.1. *Let V be a reflexive Banach, $V_0, V_{11}, \cdots, V_{1m}$ some closed subspaces of V, and K a non empty closed convex subset of V which satisfies Assumption 2.1, Assumption 2.2 when we apply Algorithms 4.1, and Assumption 2.3 in the case of Algorithms 4.2. Also, we assume that F is Gâteaux differentiable and satisfies (2.1) and (2.2) with $p = q = 2$, the functional φ is convex and lower semicontinuous in the second variable, satisfies (4.2), (4.3) for Algorithm 4.1, (4.4) for Algorithm 4.2, and $F + \varphi$ satisfies the coercivity condition (4.1) if K is not bounded. Let*

$$M = \sup\{||v|| \,:\, F(v) + \varphi(u,v) \le F(u^0) + \varphi(u,u^0)\} \tag{4.18}$$

where u is the solution of problem (4.5) and u^0 is its initial approximation in Algorithms 4.1 or 4.2. On these conditions, there exists a constant $\chi_M > 0$ and if

$$\tfrac{c_M}{\alpha_M} \le \chi_M \tag{4.19}$$

then, the norms of the approximations of the solution u of problem (4.5) obtained from these algorithms are bounded by M and we have the following error estimations:

$$\begin{aligned} & F(u^n) + \varphi(u,u^n) - F(u) - \varphi(u,u) \le \\ & (\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u,u^0) - F(u) - \varphi(u,u)], \end{aligned} \tag{4.20}$$

$$\|u^n - u\|^2 \le \tfrac{2}{\alpha_M} (\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u,u^0) - F(u) - \varphi(u,u)]. \tag{4.21}$$

The constant $C_1 > 0$ depends on the functionals F and φ, the solution u, the initial approximation u^0, m, and the constant C_0.

Remark 4.1. *For Algorithm 4.1, constant C_1 can be written as,*

$$\begin{aligned} & C_1 = C_2/C_3 \\ & C_2 = \beta_M (m+1)(1 + 2C_0 + \tfrac{C_0}{\varepsilon_1}) + \\ & c_M (m+1)(1 + 2C_0 + \tfrac{1+3C_0}{\varepsilon_2}) \\ & C_3 = \tfrac{\alpha_M}{2} - c_M (1 + \varepsilon_3)(m+1) \end{aligned} \tag{4.22}$$

where

$$\varepsilon_1 = \varepsilon_2 = \frac{2c_M(m+1)}{\frac{\alpha_M}{2} - c_M(m+1)},\ \varepsilon_3 = \frac{\frac{\alpha_M}{2} - c_M(m+1)}{2c_M(m+1)}, \tag{4.23}$$

and χ_M is the smallest positive solution of equation

$$(m+1)\chi_M + \sqrt{2(m+1)(25C_0 + 8)\frac{\beta_M}{\alpha_M}\chi_M} - \frac{1}{2} = 0. \tag{4.24}$$

Also, in the case of Algorithm 4.2, constant C_1 *can be written as,*

$$\begin{aligned} &C_1 = \tfrac{m+1-r}{r} + C_2\tfrac{m+1}{r} \\ &C_2 = \tfrac{m+1}{C_3}[\beta_M(1 + C_0(1 + \tfrac{1}{2\varepsilon_2}))+ \\ &c_M(1 + C_0 + \tfrac{1+2C_0}{2\varepsilon_3})] \\ &C_3 = \tfrac{\alpha_M}{2} - c_M(1 + \tfrac{1}{2\varepsilon_1})(m + 1) \end{aligned} \tag{4.25}$$

where

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{c_M(m+1)}{\frac{\alpha_M}{2} - c_M(m+1)}, \tag{4.26}$$

and χ_M *is the smallest positive solution of equation*

$$\begin{aligned} &(\tfrac{1}{2} - C_0\chi_M)\tfrac{\alpha_M}{\beta_M} = \\ &(1 + 3C_0)\tfrac{\chi_M(m+1)}{\frac{1}{2}-\chi_M(m+1)} + 2(1 + C_0)\tfrac{(\chi_M)^2(m+1)^2}{[\frac{1}{2}-\chi_M(m+1)]^2} \end{aligned} \tag{4.27}$$

Proof of Theorem 4.1. As for Theorem 3.1, the proof in the case of the multiplicative Algorithms 4.1 is identical with that of Theorem 2 in [7] (see also [5]), except the changes of notation due to the introduction of the convex sets K_1 and K_0, and will be omitted. Moreover, we shall prove the theorem only for Algorithm 4.2, the proof of its variant with v_{1i}^{n+1} and v_0^{n+1} in (4.17) is similar.

Step 1. Evidently, the existence of $M > 0$ satisfying (4.18) follows from the coercivity of $F + \varphi$. Now, we show that this M has the properties in the statement of the theorem. In this proof, equations (2.1), (2.2) and (4.2) will be used with u, v, v_1, v_2, w_1 and w_2 replaced only with the solution u of problem (4.5) or its approximations obtained from Algorithms 4.1, 4.2 or their variants. Let us assume that M_n is the maximum of the norms of these approximations obtained after n iterations. With this M_n, we shall get that error estimation (4.20) holds until the iteration n. Even if C_1 depends on M_n, this error estimation implies $F(u^n) + \varphi(u^n) \le F(u^0) + \varphi(u^0)$. Moreover, using the minimization problems equivalent with the inequalities in the algorithms we get that the other approximations of u satisfy a similar equation, ie. $M_n \le M$.

Step 2. From (4.14), (4.15) and (2.4), we get that, for any $n \ge 0$ and $i = 1, \cdots, m$,

$$\begin{aligned} &F(u^n) - F(u^n + w_{1i}^{n+1}) + \varphi(v_{1i}^{n+1}, u^n)- \\ &\varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1}) \ge \tfrac{\alpha_M}{2}||w_{1i}^{n+1}||^2, \\ &F(u^n) - F(u^n + w_0^{n+1}) + \varphi(v_0^{n+1}, u^n)- \\ &\varphi(v_0^{n+1}, u^n + w_0^{n+1}) \ge \tfrac{\alpha_M}{2}||w_0^{n+1}||^2 \end{aligned} \tag{4.28}$$

Also, in view of (4.7), we get

$$\begin{aligned} &F(u^n + w_{1i}^{n+1}) - F(u) + \varphi(u, u^n + w_{1i}^{n+1})- \\ &\varphi(u, u) \ge \tfrac{\alpha_M}{2}\|u^n + w_{1i}^{n+1} - u\|^2 \\ &F(u^n + w_0^{n+1}) - F(u) + \varphi(u, u^n + w_0^{n+1})- \\ &\varphi(u, u) \ge \tfrac{\alpha_M}{2}\|u^n + w_0^{n+1} - u\|^2 \end{aligned} \tag{4.29}$$

for $n \geq 0$ and $i = 1, \cdots, m$. From (3.22) and (4.28), we have

$$\begin{aligned}
&F(u^{n+1}) \leq (1-r)F(u^n) + \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m F(u^n + w_{1i}^{n+1}) + F(u^n + w_0^{n+1})] \leq \\
&F(u^n) - \tfrac{r}{m+1}\tfrac{\alpha_M}{2}[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^2 + ||w_0^{n+1}||^2] + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(v_{1i}^{n+1}, u^n) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1})) + \\
&\varphi(v_0^{n+1}, u^n) - \varphi(v_0^{n+1}, u^n + w_0^{n+1})]
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\tfrac{r}{m+1}\tfrac{\alpha_M}{2}[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^2 + ||w_0^{n+1}||^2] \leq F(u^n) - F(u^{n+1}) + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(v_{1i}^{n+1}, u^n) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1})) + \\
&\varphi(v_0^{n+1}, u^n) - \varphi(v_0^{n+1}, u^n + w_0^{n+1}))]
\end{aligned} \tag{4.30}$$

Using (4.2) and the convexity of φ in the second variable, we have

$$\begin{aligned}
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(v_{1i}^{n+1}, u^n) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1})) + \\
&\varphi(v_0^{n+1}, u^n) - \varphi(v_0^{n+1}, u^n + w_0^{n+1})] - \varphi(u, u^n) + \varphi(u, u^{n+1}) \leq \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(v_{1i}^{n+1}, u^n) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1})) + \\
&\varphi(v_0^{n+1}, u^n) - \varphi(v_0^{n+1}, u^n + w_0^{n+1})] + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u, u^n + w_{1i}^{n+1}) - \varphi(u, u^n)) + \\
&\varphi(u, u^n + w_0^{n+1}) - \varphi(u, u^n)] \leq \\
&\tfrac{r}{m+1} c_M [\textstyle\sum_{i=1}^m ||u^n + w_{1i}^{n+1} - u|| ||w_{1i}^{n+1}|| + \\
&||u^n + w_0^{n+1} - u|| ||w_0^{n+1}||] \leq \\
&\tfrac{r}{m+1} c_M [\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}|| + ||w_0^{n+1}|| + \\
&||\tilde{u}^{n+1} - u||][\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}|| + ||w_0^{n+1}||] \leq \\
&\tfrac{r}{m+1} c_M (1 + \tfrac{1}{2\varepsilon_1})(m+1)[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^2 + ||w_0^{n+1}||^2] + \\
&\tfrac{r}{m+1} c_M \tfrac{\varepsilon_1}{2} ||\tilde{u}^{n+1} - u||^2
\end{aligned} \tag{4.31}$$

for any $\varepsilon_1 > 0$, where $\tilde{u}^{n+1}$ is defined in (3.27). In view of (4.30) and (4.31), we get

$$\begin{aligned}
&[\tfrac{\alpha_M}{2} - c_M(1 + \tfrac{1}{2\varepsilon_1})(m+1)][\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^2 + ||w_0^{n+1}||^2] \leq \\
&\tfrac{m+1}{r}[F(u^n) - F(u^{n+1}) + \varphi(u, u^n) - \varphi(u, u^{n+1})] + \\
&c_M \tfrac{\varepsilon_1}{2} ||\tilde{u}^{n+1} - u||^2
\end{aligned} \tag{4.32}$$

for any $\varepsilon_1 > 0$.

Step 3. Applying Assumption 2.3 for $w = u^n$ and $v = u$, we get a decomposition u_0^n, $u_{11}^n, \cdots, u_{1m}^n$ of $u - u^n$. From Assumption 2.3, we can replace v_{1i} and v_0 by u_{1i}^n and u_0^n in (4.14) and (4.15), respectively, and in view of the convexity of F, (2.4), (4.14) and (4.15), we obtain

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u, u^{n+1}) - \varphi(u, u) + \tfrac{r}{m+1}\tfrac{\alpha_M}{2}||u - \tilde{u}^{n+1}||^2 \leq \\
&(1 - \tfrac{r}{m+1})[F(u^n) - F(u)] + \tfrac{r}{m+1}[F(\tilde{u}^{n+1}) - F(u) + \\
&\tfrac{\alpha_M}{2}||u - \tilde{u}^{n+1}||^2] + \varphi(u, u^{n+1}) - \varphi(u, u) \leq (1 - \tfrac{r}{m+1})[F(u^n) - F(u)] + \\
&\tfrac{r}{m+1}\langle F'(\tilde{u}^{n+1}), \tilde{u}^{n+1} - u\rangle + \varphi(u, u^{n+1}) - \varphi(u, u) \leq \\
&(1 - \tfrac{r}{m+1})[F(u^n) - F(u)] + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle + \\
&\langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle] + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(v_{1i}^{n+1}, u^n + u_{1i}^n) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1})) + \\
&\varphi(v_0^{n+1}, u^n + u_0^n) - \varphi(v_0^{n+1}, u^n + w_0^{n+1})] + \\
&\varphi(u, u^{n+1}) - \varphi(u, u)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u, u^{n+1}) - \varphi(u, u) + \\
&\tfrac{r}{m+1}\tfrac{\alpha_M}{2}||u - \tilde{u}^{n+1}||^2 \leq \\
&(1 - \tfrac{r}{m+1})[F(u^n) - F(u) + \varphi(u, u^n) - \varphi(u, u)] + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle + \\
&\langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle] + \\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(v_{1i}^{n+1}, u^n + u_{1i}^n) - \varphi(v_{1i}^{n+1}, u^n + w_{1i}^{n+1})) + \\
&\varphi(v_0^{n+1}, u^n + u_0^n) - \varphi(v_0^{n+1}, u^n + w_0^{n+1})] + \\
&\tfrac{r}{m+1}[\varphi(u, u^n) - \varphi(u, u)] + \varphi(u, u^{n+1}) - \varphi(u, u^n)
\end{aligned} \tag{4.33}$$

Using (2.2) for $p = q = 2$, Assumption 2.3 and the Hölder inequality, similarly with (3.30), we get

$$\begin{aligned}
&\textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle + \\
&\langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle \leq \\
&\beta_M(m+1)[1 + C_0(1 + \tfrac{1}{2\varepsilon_2})][\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^2 + ||w_0^{n+1}||^2] + \\
&\beta_M C_0 \tfrac{\varepsilon_2}{2}||u - \tilde{u}^{n+1}||^2
\end{aligned} \tag{4.34}$$

for any $\varepsilon_2 > 0$. Similarly with (3.22), from the convexity of φ in the second variable, we get

$$\varphi(u, u^{n+1}) \leq (1-r)\varphi(u, u^n) + \frac{r}{m+1}[\sum_{i=1}^m \varphi(u, u^n + w_{1i}^{n+1}) + \varphi(u, u^n + w_0^{n+1})]$$

Using this equation, in view of (4.4), (4.2) and Assumption 2.3, we have

$$\begin{aligned}
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m(\varphi(v_{1i}^{n+1},u^n+u_{1i}^n)-\varphi(v_{1i}^{n+1},u^n+w_{1i}^{n+1}))+\\
&\varphi(v_0^{n+1},u^n+u_0^n)-\varphi(v_0^{n+1},u^n+w_0^{n+1})]+\\
&\tfrac{r}{m+1}[\varphi(u,u^n)-\varphi(u,u)]+\varphi(u,u^{n+1})-\varphi(u,u^n)\leq\\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m(\varphi(v_{1i}^{n+1},u^n+u_{1i}^n)-\varphi(v_{1i}^{n+1},u^n+w_{1i}^{n+1}))+\\
&\varphi(v_0^{n+1},u^n+u_0^n)-\varphi(v_0^{n+1},u^n+w_0^{n+1})]+\\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m\varphi(u,u^n+w_{1i}^{n+1})+\varphi(u,u^n+w_0^{n+1})]-\\
&\tfrac{r}{m+1}[m\varphi(u,u^n)+\varphi(u,u)]\leq\\
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m(\varphi(v_{1i}^{n+1},u^n+u_{1i}^n)-\varphi(v_{1i}^{n+1},u^n+w_{1i}^{n+1}))+\\
&\varphi(v_0^{n+1},u^n+u_0^n)-\varphi(v_0^{n+1},u^n+w_0^{n+1})]+\\
&+\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m(\varphi(u,u^n+w_{1i}^{n+1})-\varphi(u,u^n+u_{1i}^n))+\\
&\varphi(u,u^n+w_0^{n+1})-\varphi(u,u^n+u_0^n)]\leq\\
&\tfrac{r}{m+1}c_M[\textstyle\sum_{i=1}^m\|u^n+w_{1i}^{n+1}-u\|\|w_{1i}^{n+1}-u_i^n\|+\\
&\|u^n+w_0^{n+1}-u\|\|w_0^{n+1}-u_0^n\|]\leq\\
&\tfrac{r}{m+1}c_M[\|\tilde{u}^{n+1}-u\|+\textstyle\sum_{i=1}^m\|w_{1i}^{n+1}\|+\|w_0^{n+1}\|]\\
&[\textstyle\sum_{i=1}^m(\|w_{1i}^{n+1}\|+\|u_i^n\|)+\|w_0^{n+1}\|+\|u_0^n\|]\leq\\
&\tfrac{r}{m+1}c_M[\|\tilde{u}^{n+1}-u\|+\textstyle\sum_{i=1}^m\|w_{1i}^{n+1}\|+\|w_0^{n+1}\|]\cdot\\
&[C_0\|\tilde{u}^{n+1}-u\|+(1+C_0)(\textstyle\sum_{i=1}^m\|w_{1i}^{n+1}\|+\|w_0^{n+1})]
\end{aligned}$$

or

$$\begin{aligned}
&\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m(\varphi(v_{1i}^{n+1},u^n+u_{1i}^n)-\varphi(v_{1i}^{n+1},u^n+w_{1i}^{n+1}))+\\
&+\varphi(v_0^{n+1},u^n+u_0^n)-\varphi(v_0^{n+1},u^n+w_0^{n+1})]+\\
&\tfrac{r}{m+1}[\varphi(u,u^n)-\varphi(u,u)]+\varphi(u,u^{n+1})-\varphi(u,u^n)\leq\\
&\tfrac{r}{m+1}c_M[C_0+(1+2C_0)\tfrac{\varepsilon_3}{2}]\|\tilde{u}^{n+1}-u\|^2+\\
&rc_M[1+C_0+\tfrac{1+2C_0}{2\varepsilon_3}][\textstyle\sum_{i=1}^m\|w_{1i}^{n+1}\|^2+\|w_0^{n+1}\|^2]
\end{aligned}\tag{4.35}$$

for any $\varepsilon_3>0$. Consequently, from (4.33)–(4.35), we have

$$\begin{aligned}
&F(u^{n+1})-F(u)+\varphi(u,u^{n+1})-\varphi(u,u)+\\
&\{\tfrac{\alpha_M}{2}-\beta_MC_0\tfrac{\varepsilon_2}{2}-c_M[C_0+(1+2C_0)\tfrac{\varepsilon_3}{2}]\}||u-\tilde{u}^{n+1}||^2\leq\\
&\tfrac{m+1-r}{r}[F(u^n)-F(u^{n+1})+\varphi(u,u^n)-\varphi(u,u^{n+1})]+\\
&(m+1)\{\beta_M[1+C_0(1+\tfrac{1}{2\varepsilon_2})]+c_M[1+C_0+\tfrac{1+2C_0}{2\varepsilon_3}]\}\cdot\\
&[\textstyle\sum_{i=1}^m||w_{1i}^{n+1}||^2+||w_0^{n+1}||^2]
\end{aligned}\tag{4.36}$$

for any ε_2, $\varepsilon_3>0$.

Step4. Writing C_1, C_2 and C_3 as in (4.25), and

$$C_4=\tfrac{\alpha_M}{2}-\beta_MC_0\tfrac{\varepsilon_2}{2}-c_M(C_0+\tfrac{1+2C_0}{2}\varepsilon_3)-c_M\tfrac{\varepsilon_1}{2}C_2$$

then, from (4.36) and (4.32), on the condition $C_3>0$, we get

$$\begin{aligned}
&F(u^{n+1})-F(u)+\varphi(u,u^{n+1})-\varphi(u,u)+C_4||u-\tilde{u}^{n+1}||^2\leq\\
&C_1[F(u^n)-F(u^{n+1})+\varphi(u,u^n)-\varphi(u,u^{n+1})]
\end{aligned}\tag{4.37}$$

Now, if $C_4 \geq 0$, then (4.20) can be obtained from (4.37). Also, in view of (4.7), (4.21) can be obtained from (4.20).

We can easily see that C_4, as a function of ε_1, ε_2, and ε_3, reaches its maximum for the values given in (4.26), and this is

$$\begin{aligned} C_{4max} = & \tfrac{\alpha_M}{2} - c_M C_0 - [\beta_M C_0 + c_M(1+2C_0)] \tfrac{c_M(m+1)}{\frac{\alpha_M}{2} - c_M(m+1)} \\ & -(1+C_0)(\beta_M + c_M) \tfrac{c_M^2 (m+1)^2}{[\frac{\alpha_M}{2} - c_M(m+1)]^2}. \end{aligned}$$

Condition $C_{4max} \geq 0$ is satisfied if

$$(\frac{1}{2} - C_0 \frac{c_M}{\alpha_M}) \frac{\alpha_M}{\beta_M} \geq (1+3C_0) \frac{\frac{c_M}{\alpha_M}(m+1)}{\frac{1}{2} - \frac{c_M}{\alpha_M}(m+1)} + 2(1+C_0) \frac{(\frac{c_M}{\alpha_M})^2 (m+1)^2}{[\frac{1}{2} - \frac{c_M}{\alpha_M}(m+1)]^2}$$

Writing $\chi_M = \frac{c_M}{\alpha_M}$, we see that equation (4.27) has a solution $\chi_M \in (0, \frac{1}{2C_0})$, and if it is the smallest one and we take $\frac{c_M}{\alpha_M} \leq \chi_M$, then $C_{4max} \geq 0$.

The value of C_3 for ε_1 in (4.26) is

$$C_{3\max} = \frac{1}{2}(\frac{\alpha_M}{2} - c_M(m+1)).$$

Since we can always take $C_0 \geq m+1$, the above solution χ_M of equation (4.27) satisfies $\chi_M < \frac{1}{2(m+1)}$, and therefore, we get $C_{3\max} > 0$ for any $\frac{c_M}{\alpha_M} \leq \chi_M$.

□

5 Convergence rates of the two-level methods

Algorithms in the previous section can be viewed as two-level Schwarz methods in a subspace correction variant if we use the finite element spaces. The convergence rates given in Theorems 3.1 and 4.1 depend on the functionals F and φ, the number m of the subspaces and the constant C_0 introduced in Assumption 2.2 or 2.3. In the multiplicative methods, the number of subspaces can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rates essentially depend on the constant C_0.

We prove in this section that Assumptions 2.2 and 2.3 as well as conditions (3.2), (3.3), (4.3) and (4.4) hold for closed convex sets K of two-obstacle type for which we construct level convex sets K_1 and K_2 as in Assumption 2.1. Also, we are able to explicitly write the dependence of C_0 on the domain decomposition and mesh parameters. Therefore, from Theorems 3.1 and 4.1, we can conclude that the two-level methods globally converge for variational inequalities of the second kind and quasi-variational inequalities. Moreover, the introduced methods have an optimal computing complexity per iteration, and from the dependence of C_0 on the mesh and domain decomposition parameters, the convergence rate is optimal, ie. is similar with that in the case of linear equations, for instance. This convergence rate depends very weakly on the mesh and domain decomposition parameters, and, for some particular choices, it is even independent of them.

We consider two simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ of the domain $\Omega \subset \mathbf{R}^d$ of mesh sizes h and H, respectively. The mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, and we assume that both the families, of fine and coarse meshes, are regular (see [8], p. 124, for instance). We assume that the domain Ω is decomposed as

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{5.1}$$

and that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. The overlapping parameter of this decomposition will be denoted by δ. In addition, we suppose that there exists a constant C, independent of both meshes, such that the diameter of the connected components of each Ω_i is less than CH. We point out that the domain Ω may be different from $\Omega_0 = \cup_{\tau\in\mathcal{T}_H} \tau$, but we assume that if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$ then it also lies on $\partial\Omega$, and there exists a constant C, independent of both meshes, such that $\mathrm{dist}(x, \Omega_0) \le CH$ for any node x of $\mathcal{T}_h$.

We consider the piecewise linear finite element space

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\}, \tag{5.2}$$

and also, for $i = 1, \ldots, m$, let

$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega\backslash\Omega_i\} \tag{5.3}$$

be the subspaces of V_h corresponding to the domain decomposition $\Omega_1, \ldots, \Omega_m$. We also introduce the continuous, piecewise linear finite element space corresponding to the H-level,

$$V_H^0 = \{v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0\}, \tag{5.4}$$

where the functions v are extended with zero in $\Omega\backslash\Omega_0$. The spaces V_h and V_h^i, $i = 1, \ldots, m$, and V_H^0 are considered as subspaces of $W^{1,s}$, for some fixed $1 \le s \le \infty$. We denote by $\|\cdot\|_{0,s}$ the norm in L^s, and by $\|\cdot\|_{1,s}$ and $|\cdot|_{1,s}$ the norm and seminorm in $W^{1,s}$, respectively.

We consider problems (3.4) and (4.5) in the space $V = V_h$ with the convex set of the form

$$K = \{v \in V_h : \varphi \le v \le \psi\}, \tag{5.5}$$

where $\varphi,\ \psi \in V_h$, $\varphi \le \psi$. The two-level methods are obtained from the algorithms in the previous sections with $V_0 = V_H^0$, $V_{11} = V_h^1, \ldots, V_{1m} = V_h^m$.

Naturally, if for the previous choice of the convex set K and the subspaces $V_0,\ V_{11}, \ldots, V_{1m}$ of V, we construct the convex sets K_1 and K_0 as in Assumptions 2.1 and prove that Assumptions 2.2 and 2.3 are satisfied, we can conclude that these algorithms converge if we prove in addition that the functionals φ satisfy, depending on problem and on the utilized algorithm, (3.2), (3.3), (4.3) or (4.4).

In general, the functionals φ in the original problem do not satisfy these technical conditions. For this reason, they have been replaced [7] by approximations, obtained by numerical quadrature in V_h. In the case of the variational inequalities of the second kind, we assume that the functional φ is of the form

$$\varphi(v) = \sum_{\kappa \in \mathcal{N}_h} s_\kappa(h)\phi(v(x_\kappa)) = \sum_{k \in \mathcal{N}_h} s_\kappa(h)\phi_\kappa(v) \tag{5.6}$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is a continuous and convex function, $\mathcal{N}_h$ is the set of nodes of the mesh partition $\mathcal{T}_h$, and $s_\kappa(h) \ge 0$, $\kappa \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. For the ease of notation, we have written $\phi_\kappa(v) = \phi(v(x_\kappa))$. We see that ϕ_κ, $\kappa \in \mathcal{N}_h$, can be viewed as some functionals $\phi_\kappa : V_h \to \mathbf{R}$ which satisfy

$$\phi_\kappa(L_h(\theta v + (1-\theta)w)) \le \theta(x_\kappa)\phi_\kappa(v) + (1-\theta(x_\kappa))\phi_\kappa(w) \tag{5.7}$$

for any $v, w \in K$, and any function $\theta : \bar{\Omega} \to \mathbf{R}$ which satisfy $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$. Above, we have denoted by L_h the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_h$.

For the quasi-variational inequalities, we assume that the functional φ is of the form

$$\varphi(u, v) = \sum_{\kappa \in \mathcal{N}_h} s_\kappa(h)\phi(u, v(x_\kappa)) = \sum_{k \in \mathcal{N}_h} s_\kappa(h)\phi_\kappa(u, v) \tag{5.8}$$

where $\phi : V_h \times \mathbf{R} \to \mathbf{R}$ is continuous, and, as above, $s_\kappa(h) \ge 0$, $\kappa \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. Also, we assume that $\varphi(u, \cdot) : \mathbf{R} \to \mathbf{R}$ is convex for any $u \in V_h$, and, for the ease of notation, we have written $\phi_\kappa(u, v) = \phi(u, v(x_\kappa))$. We see that ϕ_κ, $\kappa \in \mathcal{N}_h$, can be viewed as some functionals $\phi_\kappa : V_h \times V_h \to \mathbf{R}$ which satisfy

$$\phi_\kappa(u, L_h(\theta v + (1-\theta)w)) \le \theta(x_\kappa)\phi_\kappa(u, v) + (1-\theta(x_\kappa))\phi_\kappa(u, w) \tag{5.9}$$

for any $u \in V_h$, $v, w \in K$, and any function $\theta : \bar{\Omega} \to \mathbf{R}$ with the properties $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$.

To verify that Assumptions 2.1–2.3 for the convex set in (5.5), we consider the operator $I_H : V_h \to V_H^0$, which has been introduced in [3], and we recall here some of its properties. For any $x \in \Omega$, we have

$$\begin{array}{ll} & 0 \le I_H v(x) \le v(x) \text{ if } v(x) > 0, \\ (5.10) & 0 \ge I_H v(x) \ge v(x) \text{ if } v(x) < 0, \\ & I_H v = 0 \text{ on } \tau \in \mathcal{T}_H \text{ if there exists a } x \in \tau \text{ such that } v(x) = 0 \end{array}$$

for any $v \in V_h$. Consequently, writing

$$\theta_v(x) = \begin{cases} \frac{I_H v(x)}{v(x)} & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0, \end{cases}$$

then $\theta_v \in C^0(\bar{\Omega})$, $\theta_v|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, $0 \le \theta_v \le 1$, and

$$I_H v = \theta_v v \tag{5.11}$$

for any $v \in V_h$. Also, I_H has the following properties (see Lemma 4.3 in [3]) for any $v \in V_h$:

$$\|I_H v - v\|_{0,s} \le C H C_{d,s}(H,h) |v|_{1,s} \tag{5.12}$$

$$\|I_H v\|_{0,s} \le C \|v\|_{0,s} \text{ and } |I_H v|_{1,s} \le C C_{d,s}(H,h) |v|_{1,s}, \tag{5.13}$$

where

$$C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or} \\ & 1 \le d < s \le \infty \\ (\ln \frac{H}{h} + 1)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ (\frac{H}{h})^{\frac{d-s}{s}} & \text{if } 1 \le s < d < \infty, \end{cases} \tag{5.14}$$

Now, we define the level convex sets K_1 and K_0, satisfying Assumption 2.1. Let K be the convex set defined in (5.5), and a $w \in K$. We consider

$$\begin{array}{ll} & K_1 = [\varphi_1, \psi_1],\ \varphi_1 = \varphi - w,\ \psi_1 = \psi - w, \\ (5.15) & K_0 = [\varphi_0, \psi_0],\ \varphi_0 = I_H(\varphi_1 - w_1),\ \psi_0 = I_H(\psi_1 - w_1) \end{array}$$

where w_1 has been chosen in K_1. Using (5.10) we can easily prove

Proposition 5.1. *Assumption 2.1 holds for the convex sets K_1 and K_0 defined in (5.15) for any $w \in K$ and $w_1 \in K_1$.*

Now, let us consider $u, w \in K$ and define

$$u_1 = u - w - I_H(u - w - w_1) \text{ and } u_0 = I_H(u - w - w_1). \tag{5.16}$$

where $w_1 \in K_1$. Now, we prove

Lemma 5.1. *If K_1 and K_2 are defined in (5.15), and u_1 and u_2 are defined in (5.16), then*

$$u_1 \in K_1,\ u_0 \in K_0 \text{ and } u - w = u_1 + u_0 \tag{5.17}$$

and

$$\begin{aligned} &|u_0|_{1,s},\ |u_1|_{1,s} \leq CC_{d,s}(H,h)[|w_1|_{1,s} + |u-w|_{1,s}], \\ &||u_1||_{0,s} \leq ||w_1||_{0,s} + CHC_{d,s}(H,h)[|w_1|_{1,s} + |u-w|_{1,s}] \\ &||u_0||_{0,s} \leq C[||u-w||_{0,s} + ||w_1||_{0,s}]. \end{aligned} \tag{5.18}$$

Proof. Since $u - w, w_1 \in K_1$, using (5.11), we get $u_1 = u - w - \theta_{u-w-w_1}(u - w - w_1) = (1 - \theta_{u-w-w_1})(u - w) + \theta_{u-w-w_1} w_1 \in K_1$. Also, since $u, w + w_1 \in K$, we have $w + w_1 + u_0 = w + w_1 + \theta_{u-w-w_1}(u - w - w_1) = (1 - \theta_{u-w-w_1})(w + w_1) + \theta_{u-w-w_1} u \in K$, ie. $u_0 \in K_0$. Evidently, $u - w = u_1 + u_0$, and therefore (5.17) holds. Inequalities (5.18) easily follows from (5.12) and (5.13). □

To prove that Assumption 2.2 holds, we associate to the decomposition (5.1) of Ω some functions $\theta_i \in C(\bar{\Omega}_i)$, $\theta_i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$, such that

$$\begin{aligned} &0 \leq \theta_i \leq 1 \text{ on } \Omega,\ \theta_i = 0 \text{ on } \cup_{j=i+1}^m \Omega_j \backslash \Omega_i \text{ and} \\ &\theta_i = 1 \text{ on } \Omega_i \backslash \cup_{j=i+1}^m \Omega_j. \end{aligned} \tag{5.19}$$

Such functions θ_i with the above properties have been introduced in [1] and they are constructed using unity partitions of the domains $\cup_{j=i}^m \Omega_j$, $i = 1, \ldots, m$. Also, to prove that Assumption 2.3 holds, we associate to the decomposition (5.1), a unity partition $\{\theta_i\}_{1 \leq i \leq m}$, with $\theta_i \in C^0(\bar{\Omega})$, $\theta_i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$,

$$0 \leq \theta_i \leq 1 \text{ on } \Omega,\ \text{supp } \theta_i \subset \overline{\Omega}_i \text{ and } \textstyle\sum_{i=1}^m \theta_i = 1 \tag{5.20}$$

Since the overlapping size of the domain decomposition is δ, the functions θ_i in (5.19) and (5.20) can be chosen to satisfy

$$|\partial_{x_k} \theta_i| \leq C/\delta, \text{ a.e. in } \Omega, \text{ for any } k = 1, \ldots, d \tag{5.21}$$

As in (5.21), we denote in the following by C a generic constant which does not depend on either the mesh or the decomposition of the domain.

Now, we prove the following proposition which, in particular, shows that the constant C_0 in Assumptions 2.2 and 2.3 is independent of the mesh and domain decomposition parameters if H/δ and H/h are kept constant when $h \to 0$. Also, we can conclude from this proposition that the error estimations in Theorems 3.1 and 4.1 hold for the two-level methods, arising from the algorithms introduced in the previous section, for variational and quasi-variational inequalities of the second kind.

Proposition 5.2. *Assumptions 2.2 and 2.3 holds for the convex sets* K_1 *and* K_0 *defined in (5.15) with the constant* C_0 *written as*

$$C_0 = C(m+1)C_{d,s}(H,h)[1+(m-1)\tfrac{H}{\delta}] \tag{5.22}$$

where C *is independent of the mesh and domain decomposition parameters, and* $C_{d,s}(H,h)$ *is given in* (5.14). *Also, conditions (3.2) and (3.3), for functionals* φ *of the form (5.6), and (4.4) and (4.3), for the functionals* φ *in (5.8), are satisfied.*

Proof. For Assumption 2.2, let us consider $u,\ w \in K$ and $w_{1i} \in V_h^i$ such that $w_{11} + \ldots + w_{1i} \in K_1,\ i = 1, \ldots, m$. In the construction of the convex sets K_0, we take $w_1 = \sum_{i=1}^m w_{1i}$, and consider u_1 and u_0 given in (5.16). Now, we define

$$\begin{aligned} &u_{11} = L_h(\theta_1 u_1 + (1-\theta_1)w_{11}) \text{ and} \\ &u_{1i} = L_h(\theta_i(u_1 - \textstyle\sum_{j=1}^{i-1} u_{1j}) + (1-\theta_i)w_{1i}), \quad i = 2, \ldots, m, \end{aligned} \tag{5.23}$$

with θ_i in (5.19), L_h being the P_1-Lagrangian interpolation. Like in Proposition 3.1 in [3] (see also [1] or [7]), where we take $v = u_1$ and $w = 0$, we can prove that

$$\begin{aligned} &u_{1i} \in V_h^i,\ w_{11} + \ldots + w_{1i-1} + u_{1i} \in K_1,\ i = 1, \ldots, m \text{ and} \\ &u_1 = \textstyle\sum_{i=1}^m u_{1i}. \end{aligned} \tag{5.24}$$

From (5.17) and (5.24), we get that the first two conditions of Assumption 2.2 are satisfied. We estimate now the constant C_0. In view of (5.23), and using (5.21) and some proprieties of the Lagrange interpolation operator, as in the proof of Proposition 3.1 in [3], we can write

$$\begin{aligned} &||u_{1i}||_{1,s} \le C(|u_1|_{1,s} + (1 + \tfrac{m-1}{\delta})||u_1||_{0,s} + \\ &(1 + (m-1)\tfrac{H}{\delta}) \textstyle\sum_{j=1}^m |w_{1j}|_{1,s}). \end{aligned} \tag{5.25}$$

In view of (5.18) and (5.25), we have

$$\begin{aligned} &||u_{1i}||_{1,s} \le CC_{d,s}(H,h)[1 + (m-1)\tfrac{H}{\delta}][|u-w|_{1,s} + \textstyle\sum_{j=1}^m |w_{1j}|_{1,s}], \\ &i = 1, \ldots, m, \text{ and } ||u_0||_{1,s} \le CC_{d,s}(H,h)[||u-w||_{1,s} + \textstyle\sum_{j=1}^m |w_{1j}|_{1,s}] \end{aligned}$$

ie. Assumption 2.2 is satisfied with C_0 given in (5.22).

In the case of Assumption 2.3, we consider $u, w \in K$ and construct u_0 and u_1 as in (5.16) with $w_1 = 0$. Then, we define

$$u_{1i} = L_h(\theta^i u_1), \quad i = 1, \dots, m, \tag{5.26}$$

and it is clear that $u_{1i} \in V_h^i \cap K_1$ and the first condition in Assumption 2.3 is satisfied. As above, we get

$$\begin{array}{l} ||u_{1i}||_{1,s} \leq C C_{d,s}(H,h)[1 + (m-1)\frac{H}{\delta}]|u - w|_{1,s}, \ i = 1, \dots, m, \text{ and} \\ ||u_0||_{1,s} \leq C C_{d,s}(H,h)||u - w||_{1,s} \end{array}$$

ie. Assumption 2.3 is satisfied with C_0 given in (5.22).

Finally, to prove that conditions (3.2), (3.3), (4.3) and (4.4) hold for the functionals φ in (5.8) and (5.6), it is sufficient to show that these conditions are true for each φ_κ, $\kappa \in \mathcal{N}_h$. We prove in the following only condition (3.2), the proof for the other ones being similar. Like in the proof of Propositions 1 and 2 in [7], using (5.16), (5.23) and (5.11), in view of (5.7), we have

$$\begin{array}{l} \sum_{i=1}^m [\varphi_\kappa(w + \sum_{j=1}^{i-1} w_{1j} + u_{1i}) - \varphi_\kappa(w + \sum_{j=1}^{i-1} w_{1j} + w_{1i})] + \\ \varphi_\kappa(w + w_1 + u_0) - \varphi(w + w_1 + w_0) \leq \\ \varphi_\kappa(w + u_1) - \varphi_\kappa(w + w_1) + \varphi_\kappa(w + w_1 + u_0) - \varphi(w + w_1 + w_0) = \\ \varphi_\kappa(u - I_H(u - w - w_1)) - \varphi_\kappa(w + w_1) + \\ \varphi_\kappa(w + w_1 + I_H(u - w - w_1)) - \varphi(w + w_1 + w_0) \leq \\ (1 - \theta_{u-w-w_1}(x_\kappa))\varphi_\kappa(u) + \theta_{u-w-w_1}(x_\kappa)\varphi_\kappa(w + w_1) - \varphi_\kappa(w + w_1) + \\ (1 - \theta_{u-w-w_1}(x_\kappa))\varphi_\kappa(w + w_1) + \theta_{u-w-w_1}(x_\kappa)\varphi_\kappa(u) - \varphi(w + w_1 + w_0) = \\ \varphi_\kappa(u) - \varphi(w + \sum_{i=1}^m w_{1i} + w_0) \end{array}$$

ie. (3.2) holds. □

Remark 5.1. In this Section 5, we have assumed that, in the case of the quasi-variational inequalities, the functional φ is of the form (5.8). We notice that the proofs of Proposition 5.2 also holds if we replace the functional $\varphi(u, v)$ in (5.8) with

$$\varphi(u, v) = \sum_{\kappa \in \mathcal{N}_h} s_\kappa(h) \phi(u(x_\kappa), v(x_\kappa)) = \sum_{k \in \mathcal{N}_h} s_\kappa(h) \phi_\kappa(u, v) \tag{5.27}$$

where $s_\kappa(h) \geq 0$, and $\phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous and convex in the second variable. We have denoted above $\phi_\kappa(u, v) = \phi(u(x_\kappa), v(x_\kappa))$, $\kappa \in \mathcal{N}_h$. In general, (5.6), (5.8) or (5.27) represent numerical approximations of some integrals. Concerning to condition (4.2) imposed on φ of the form (5.8) or (5.27), in the case of quasi-variational inequalities, we have to check it for each particular problem we solve.

The results of this section have referred to problems in $W^{1,s}$ with Dirichlet boundary conditions. We point out that similar results can be obtained for problems in $(W^{1,s})^d$ or problems with mixed boundary conditions.

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References

[1] L. Badea, On the Schwarz alternating method with more than two subdomains for nonlinear monotone problems, *SIAM J. Numer. Anal.*, **28**, 1, 1991, pp. 179-204.

[2] L. Badea, Convergence rate of a multiplicative Schwarz method for strongly nonlinear inequalities, in: *Analysis and optimization of differential systems*, V.Barbu, I. Lasiecka, D. Tiba and C. Varsan (Eds.), Kluwer Academic Publishers, Boston/Dordrecht/London, 2003, pp. 31-42, (also available from http://www.imar.ro/ lbadea).

[3] L. Badea, Convergence rate of a Schwarz multilevel method for the constrained minimization of non-quadratic functionals, *SIAM J. Numer. Anal.*, **44**, 2, 2006, pp. 449–477.

[4] L. Badea, Additive Schwarz method for the constrained minimization of functionals in reflexive Banach spaces, in U. Langer et al. (eds.), *Domain decomposition methods in science and engineering XVII*, LNSE 60, Springer, 2008, p. 427-434.

[5] L. Badea and R. Krause, One- and two-level multiplicative Schwarz methods for variational and quasi-variational inequalities of the second kind: Part I - general convergence results, *INS Preprint, no. 0804, Institute for Numerical Simulation, University of Bonn*, June 2008.

[6] L. Badea and R. Krause, One- and two-level multiplicative Schwarz methods for variational and quasi-variational inequalities of the second kind: Part II - frictional contact problems, *INS Preprint, no. 0805, Institute for Numerical Simulation, University of Bonn*, June 2008.

[7] L. Badea and R. Krause, One- and two-level Schwarz methods for inequalities of the second kind and their application to frictional contact problems, *Numer. Math.*, **120**, 4, 2012, pp. 573-599.

[8] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[9] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland, Amsterdam, 1976.

[10] E. Gelman and J. Mandel, On multilevel iterative method for optimization problems, *Math. Program.*, **48**, 1990, 1-17.

[11] R. Glowinski, *Numerical methods for nonlinear variational problems*, Series in Computational Physics, New York, 1984.

[12] R. Glowinski, G. H. Golub, G. A. Meurant and J. Périeux (Eds.), *First Int. Symp. on Domain Decomposition Methods*, SIAM, Philadelphia, 1988.

[13] R. Glowinski, J. L. Lions and R. Trémolières, *Analyse numérique des inéquations variationnelles*, Dunod, 1976.

[14] R. Kornhuber, Monotone multigrid methods for elliptic variational inequalities I, *Numer. Math.*, **69**, 1994, pp. 167-184.

[15] R. Kornhuber, Monotone multigrid methods for elliptic variational inequalities II, *Numer. Math.*, **72**, 1996, pp. 481-499.

[16] R. Kornhuber, *Adaptive monotone multigrid methods for nonlinear variational problems*, Teubner-Verlag, 1997.

[17] J. Mandel, A multilevel iterative method for symmetric, positive definite linear complementary problems, *Appl. Math. Opt.*, **11**, 1984, 77-95.

[18] J. Mandel, Etude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre, *C. R. Acad. Sci.*, Ser. I, **298**, 1984, 469-472.

[19] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford Science Publications, 1999.

[20] B. F. Smith, P. E. Bjørstad, and William Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Differential Equations*, Cambridge University Press, 1996.

[21] A. Toselli and O. Widlund, *Domains Decomposition Methods - Algorithms and Theory*, Springer-Verlag, Berlin, 2005.