Global convergence rate of a standard multigrid method for variational inequalities

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Complete List of Authors:	Badea, Lori; Institute of Mathematics of the Romanian Academy
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L. Badea*

Abstract

We introduce a multigrid algorithm for variational inequalities whose constraints are of the two-obstacle type. This algorithm is described as a V-cycle multigrid method, its iterations having an optimal computing complexity, but the results also hold for other types of iterations, W-cycles, for instance. In the case of the one-obstacle problems, the algorithm reduces to that introduced by Mandel in 1984 for complementarity problems and named later by Kornhuber as standard monotone multigrid method. First, we introduce the method as a subspace correction algorithm in a reflexive Banach space, prove its global convergence and estimate the error making some assumptions. By introducing the finite element spaces, this algorithm becomes a multilevel or multigrid method. In this case, we prove that the assumptions we made in the general theory are satisfied and write the convergence rate depending on the number of levels. Finally, we compare our results with the estimations of the asymptotic convergence rate existing in the literature for complementarity problems.

Keywords: domain decomposition methods, multigrid and multilevel methods, variational inequalities, nonlinear obstacle problems.
AMS subject classification: 65N55, 65K15, 65N30

1 Introduction

The first globally convergent multigrid method for variational inequalities has been proposed by Mandel in [18], [19] and [8] for complementarity problems. This method has an optimal computing complexity of iterations, ie. it is linear with respect to the degrees of freedom of the problem. It is proved in [18] that the method is globally convergent,

*Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania (e-mail: lori.badea@imar.ro)

some generalizations of the method have been given in [8], and in [19], an upper bound of the asymptotic convergence rate is given for the two-level method. Related methods have been introduced by Brandt and Cryer in [6] and Hackbush and Mittelmann in [11]. The method introduced by Mandel has been studied later by Kornhuber in [13], named standard multigrid method, and extended to variational inequalities of the second kind in [14] and [15]. A variant of this method using truncated nodal basis functions has been introduced by Hoppe and Kornhuber in [12] and analyzed by Kornhuber and Yserentant in [17]. Also, versions of this method have been applied to Signorini's problem in elasticity by Kornhuber and Krause in [16] and Wohlmuth and Krause in [23]. Evidently, the above list of citations is not exhaustive and, for further information, we can see the review article [10] written by Gräser and Kornhuber.

Regarding the convergence study of the method, an asymptotic convergence rate of $1 - 1/(1 + CJ^3)$, J being the number of levels, has been proved by Kornhuber in [13] for the complementarity problem in the bidimensional space. For the two-level method, global convergence rates have been established by Badea, Tai and Wang in [2], and for its additive variant by Badea in [4]. Also, a global convergence rate has been estimated by Tai in [20] for a multilevel subset decomposition method. In [3], we have introduced a projected multilevel method for constrained minimization problems where the convex set can be more general than of one- or two-obstacle type, for instance. The main drawback of this method, is its sub-optimal computing complexity of the iteration steps because the convex set, which is defined on the finest mesh, is used in the smoothing steps on the coarse levels.

The multigrid method introduced in the present paper is given for two-obstacle problems and its iterations have an optimal computing complexity. It is a standard V-cycle multigrid iteration, but, the presented results also hold for other types of iterations, W-cycles, for instance. In the case of the one-obstacle problems, the algorithm reduces to that introduced by Mandel for complementarity problems.

The paper is organized as follows. First, in Section 2, we introduce the method as a subspace correction algorithm in a reflexive Banach space, prove its global convergence and estimate the error making some assumptions. By introducing the finite element spaces, this algorithm becomes a multilevel or multigrid method. In Section 3 we show that this algorithm can be viewed as a multilevel method if we associate finite element spaces to the level meshes and consider decompositions of the domain at each level. We prove that the assumptions made in the previous section hold for convex sets of two-obstacle type. If the decompositions of the domain are made using the supports of the nodal basis functions we get, in Section 4, the multigrid method and write its convergence rate depending on the number of levels. We prove, for instance, that, in $\mathbf{R}^2$, it has a global convergence rate of $1 - 1/(1 + CJ^3)$, like the asymptotic convergence rate existing in the literature for complementarity problems.

For the simplicity of presentation, the convergence results of the multigrid method proposed in this paper are given only for the problems with the most applications, i.e. the problems in H^1. With small modifications, the proofs presented here can be extended to problems in $W^{1,s}$, $1 < s < \infty$, (see [5]).

2 Abstract convergence results

We consider a reflexive Banach space V and let $K \subset V$ be a nonempty closed convex set. Let $F : V \to \mathbf{R}$ be a Gâteaux differentiable functional, which is assumed to be coercive on K, in the sense that $F(v) \to \infty$, as $||v|| \to \infty$, $v \in K$, if K is not bounded. Also, we assume that there exist two constants $\alpha,\ \beta > 0$ for which

$$\begin{aligned} &\alpha||v-u||^2 \le \langle F'(v) - F'(u), v-u\rangle \\ &\text{and } ||F'(v) - F'(u)||_{V'} \le \beta||v-u||, \end{aligned} \tag{2.1}$$

for any $u, v \in V$. Above, we have denoted by F' the Gâteaux derivative of F, and V' is the dual space of V. It is evident that if (2.1) holds, then

$$\alpha||v-u||^2 \le \langle F'(v) - F'(u), v-u\rangle \le \beta||v-u||^2,$$

for any $u, v \in V$. Following the way in [9], we can prove that

$$\begin{aligned} &\langle F'(u), v-u\rangle + \tfrac{\alpha}{2}||v-u||^2 \le F(v) - F(u) \le \\ &\langle F'(u), v-u\rangle + \tfrac{\beta}{2}||v-u||^2, \end{aligned} \tag{2.2}$$

for any $u, v \in V$. We point out that since F is Gâteaux differentiable and satisfies (2.1), then F is a convex functional (see Proposition 5.5 in [7], pag. 25).

Now, let us assume that we have J closed subspaces of V, $V_1, \ldots, V_J$, and let V_{ji}, $i = 1, \ldots I_j$ be some closed subspaces of V_j, $j = J, \ldots, 1$. The subspaces V_j, $j = J, \ldots, 1$, will be associated with the grid levels, and, for each level $j = J, \ldots, 1$, V_{ji}, $i = 1, \ldots I_j$, will be associated with a domain decomposition. Let us write $I = \max_{j=J,\ldots,1} I_j$.

We consider the variational inequality

$$u \in K : \langle F'(u), v-u\rangle \ge 0, \text{ for any } v \in K, \tag{2.3}$$

and since the functional F is convex and differentiable, it is equivalent with the minimization problem

$$u \in K : F(u) \le F(v), \text{ for any } v \in K. \tag{2.4}$$

We can use, for instance, Proposition 1.2 in [7], page 34, to prove that problem (2.4) has a unique solution if F has the above properties. In view of (2.2), if $u \in K$ is the solution of problem (2.3), then

$$\frac{\alpha}{2}||v-u||^2 \le F(v) - F(u) \text{ for any } v \in K. \tag{2.5}$$

To introduce the algorithm, we make an assumption on choice of the convex sets $\mathcal{K}_j$, $j = 1, \ldots, J$, where we look for the level corrections. The chosen level convex sets depend on the current approximation in the algorithms.

Assumption 2.1. *For a given $w \in K$, we recursively introduce the convex sets $\mathcal{K}_j$, $j = J,\ J-1,\ \ldots, 1$, as*

- at level J: we assume that $0 \in \mathcal{K}_J$, $\mathcal{K}_J \subset \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in \mathcal{K}_J$,

- at a level $J-1 \ge j \ge 1$: we assume that $0 \in \mathcal{K}_j$ and $\mathcal{K}_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$, and consider a $w_j \in \mathcal{K}_j$.

We can easily check that if we take, for $j = J-1, \ldots, 1$,

$$\mathcal{K}_j \subset \{v_j \in V_j : w_{j+1} + v_j \in \mathcal{K}_{j+1}\}, \tag{2.6}$$

then $\mathcal{K}_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$.

As we will see in this section, the proposed algorithm is convergent for any level convex sets $\mathcal{K}_j$ having the properties in the above assumption. We can take, like in [3], for instance, the largest convex sets, $\mathcal{K}_j = \{v_j \in V_j : w + w_J + \ldots + w_{j-1} + v_j \in K\}$, $j = 1, \ldots, J$. In this case, we have to use the definition of K, which lies on the finest level, J, to see if the elements $v_j \in V_j$ belong to $\mathcal{K}_j$ for the coarse levels $j = 1, \ldots, J-1$. As we have already said, this leads to a sub-optimal computing complexity of the iterations. If we take $\mathcal{K}_J = \{v_J \in V_J : w + v_J \in K\}$ and $\mathcal{K}_j = \{0\}$, $j = J-1, \ldots, 1$, we get the one-level variant of the algorithm. In this case, we have a poor convergence. Evidently, a good construction of these level convex sets would be that in which the definition of K is not directly utilized in the smoothing steps, but, at the same time, the algorithm has a good convergence rate. In the case of the finite element spaces, iteration steps with optimal computing complexity will be obtained when the sets $\mathcal{K}_j$ are defined by some properties which should be verified only to the mesh nodes of the level j. In the next section, for the two-obstacle convex sets, $K = [\varphi, \psi]$, $\varphi, \psi \in V$, we construct level convex sets having the same form, $\mathcal{K}_j = [\varphi_j, \psi_j]$, $\varphi_j, \psi_j \in V_j$, $j = 1, \ldots, J$, and consequently, the computing complexity of each iteration step is optimal. Also, as we said in the introduction, the algorithm has a very good global convergence rate.

We first introduce the algorithm

Algorithm 2.1. *We start the algorithm with an arbitrary* $u^0 \in K$. *Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$, *we successively perform the following steps:*

- at the level J, *as in Assumption 2.1, with* $w = u^n$, *we construct the convex set* $\mathcal{K}_J$. *Then, we first write* $w_J^n = 0$, *and, for* $i = 1, \ldots, I_J$, *we successively calculate* $w_{Ji}^{n+1} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1} \in \mathcal{K}_J$, *the solution of the inequalities*

$$\langle F'(u^n + w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0, \tag{2.7}$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + v_{Ji} \in \mathcal{K}_J$, *and write* $w_J^{n+\frac{i}{I_J}} = w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}$,

- at a level $J-1 \geq j \geq 1$, *as in Assumption 2.1, we construct the convex set* $\mathcal{K}_j$ *with* $w = u^n$ *and* $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$. *Then, we write* $w_j^n = 0$, *and for* $i = 1, \ldots, I_j$, *we successively calculate* $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in \mathcal{K}_j$, *the solution of the inequalities*

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0, \tag{2.8}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$, *and write* $w_j^{n+\frac{i}{I_j}} = w_J^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$,

- we write $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

As inequality (2.3), inequalities (2.7) and (2.8) are equivalent with minimization problems. In order to prove the convergence of the above algorithm, we shall make two new assumptions.

Classical Cauchy-Schwarz inequality can be strengthened in certain cases when we use the multilevel decompositions, and it allows us to get some inequalities whose constants do not depend on the number of levels J. In this sense we make the following assumption.

Assumption 2.2. *1. There exist some constants* $0 < \beta_{jk} \leq 1$, $\beta_{jk} = \beta_{kj}$, $j,\ k = J, \ldots, 1$, *such that*

$$\langle F'(v + v_{ji}) - F'(v), v_{kl} \rangle \leq \beta \beta_{jk} ||v_{ji}|| ||v_{kl}||, \tag{2.9}$$

for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$, $i = 1, \ldots, I_j$ *and* $l = 1, \ldots, I_k$.

2. There exists a constant C_1 *such that*

$$|| \sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji} || \leq C_1 (\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^2)^{\frac{1}{2}}, \tag{2.10}$$

for any $w_{ji} \in V_{ji}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$.

Evidently, in view of the second equation in (2.1), inequality (2.9) holds for

$$\beta_{jk} = 1,\ j,\ k = J, \dots, 1. \tag{2.11}$$

Also, constant C_1 can be taken of the form

$$C_1 = (IJ)^{\frac{1}{2}}, \tag{2.12}$$

but, as we mentioned above, better estimations are available in the case of the multigrid decompositions. We also point out that similar inequalities with (2.9) and (2.10) have played an important role in [24] and [26] where the convergence of the multigrid method in the linear case is proved by using a spectral theory.

The second new assumption refers to additional properties asked to the convex sets $\mathcal{K}_j$, $j = 1, \dots, J$, introduced in Assumption 2.1.

ASSUMPTION 2.3. *There exists a constant $C_2 > 0$ such that for any $w \in K$, $w_{ji} \in V_{ji}$, $w_{j1} + \ldots + w_{ji} \in \mathcal{K}_j$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, and $u \in K$, there exist $u_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, which satisfy*

$$u_{j1} \in \mathcal{K}_j \text{ and } w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j,\ i = 2, \dots, I_j,\ j = J, \dots, 1, \tag{2.13}$$

$$u - w = \sum_{j=1}^{J} \sum_{i=1}^{I_j} u_{ji}, \text{ and} \tag{2.14}$$

$$\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||u_{ji}||^2 \le C_2^2 \left(||u - w||^2 + \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^2 \right). \tag{2.15}$$

The convex sets $\mathcal{K}_j$, $j = J, \dots, 1$, are constructed as in Assumption 2.1 with the above w and $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \dots, 1$.

An assumption that contains only the conditions (2.13) and (2.14), written in another form, has been introduced in [1], to prove the convergence of the Schwarz algorithm for variational inequalities. Condition (2.15) is essential in finding the convergence rate. For linear problems it has a more simple form, which does not contain the corrections w_{ji}, and has been used under similar forms in [24] and [26] (see also, [25]). This simplified condition is well-known, and concerns the stability of the decomposition of the space as a sum of subspaces. The assumption in the above form, containing the three conditions (2.13)-(2.15), has been introduced in [2] and used to prove the convergence rate of the one- and two-level methods.

The convergence result is given by

Theorem 2.1. *We consider that V is a reflexive Banach space, V_j, $j = 1, \ldots, J$, are closed subspaces of V, and V_{ji}, $i = 1, \ldots, I_j$, are closed subspaces of V_j. Also, let K be a non empty closed convex subset of V, and $\mathcal{K}_j$, $j = 1, \ldots, J$, be non empty closed subsets of V_j given by Assumption 2.1. We consider a Gâteaux differentiable functional F on V which is supposed to be coercive if K is not bounded, and which satisfies (2.1). Also, we assume that Assumptions 2.2 and 2.3 hold. On these conditions, if u is the solution of problem (2.3) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 2.1, then the following error estimations hold:*

$$F(u^n) - F(u) \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n [F(u^0) - F(u)], \tag{2.16}$$

$$||u^n - u||^2 \leq \frac{2}{\alpha}\left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n [F(u^0) - F(u)], \tag{2.17}$$

where

$$\tilde{C}_1 = \frac{1}{C_2\varepsilon}\left[1 + C_2 + C_1C_2 + \frac{C_2}{\varepsilon}\right] \tag{2.18}$$

with

$$\varepsilon = \frac{\alpha}{2\beta I(\max\limits_{k=1,\cdots,J} \sum\limits_{j=1}^{J} \beta_{kj})C_2}. \tag{2.19}$$

Proof. First, from (2.2) and inequalities (2.7) and (2.8) in which we take $v_{ji} = 0$, we have

$$\frac{\alpha}{2}||w_{ji}^{n+1}||^2 \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}}) - F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i}{I_j}}),$$

for $j = 1, \ldots, J$, $i = 1, \ldots, I_j$. Therefore, since $u^{n+1} = u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}^{n+1}$ for any $n \geq 0$, we have

$$\frac{\alpha}{2}\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^2 \leq F(u^n) - F(u^{n+1}). \tag{2.20}$$

Now, with u, the solution of problem (2.3), $w = u^n$ and $w_{ji} = w_{ji}^{n+1}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, we consider the decomposition u_{ji}, $j =$

$J, \ldots, 1$, $i = 1, \ldots, I_j$, of $u - u^n$ as in Assumption 2.3. In view of (2.2), (2.7), (2.8) and (2.9), we get

$$F(u^{n+1}) - F(u) + \frac{\alpha}{2}||u^{n+1} - u||^2 \leq \langle F'(u^{n+1}), u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}^{n+1} - u\rangle$$
$$= \sum_{j=1}^{J}\sum_{i=1}^{I_j}\langle -F'(u^n + \sum_{k=1}^{J}\sum_{l=1}^{I_k} w_{kl}^{n+1}), u_{ji} - w_{ji}^{n+1}\rangle$$
$$\leq \sum_{j=1}^{J}\sum_{i=1}^{I_j}\langle F'(u^n + \sum_{k=j+1}^{J}\sum_{l=1}^{I_k} w_{kl}^{n+1} + \sum_{l=1}^{i} w_{jl}^{n+1}) -$$
$$F'(u^n + \sum_{k=1}^{J}\sum_{l=1}^{I_k} w_{kl}^{n+1}), u_{ji} - w_{ji}^{n+1}\rangle$$
$$\leq \beta \sum_{j=1}^{J}\sum_{k=1}^{J}\beta_{kj}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||\sum_{i=1}^{I_j}||u_{ji} - w_{ji}^{n+1}||.$$

Above, we have added and subtracted the missing terms between $F'(u^n + \sum_{k=j+1}^{J}\sum_{i=1}^{I_k} w_{ki}^{n+1} + \sum_{l=1}^{i} w_{jl}^{n+1})$ and $F'(u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}^{n+1})$. Consequently, we have,

$$F(u^{n+1}) - F(u) + \frac{\alpha}{2}||u^{n+1} - u||^2 \leq$$
$$\beta I \sum_{k=1}^{J}\left(\sum_{j=1}^{J}\beta_{kj}(\sum_{i=1}^{I_j}||u_{ji} - w_{ji}^{n+1}||^2)^{\frac{1}{2}}\right)\left(\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^2\right)^{\frac{1}{2}} \leq$$
$$\beta I \left[\sum_{k=1}^{J}\left(\sum_{j=1}^{J}\beta_{kj}(\sum_{i=1}^{I_j}||u_{ji} - w_{ji}^{n+1}||^2)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}\left(\sum_{k=1}^{J}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^2\right)^{\frac{1}{2}} \leq$$
$$\beta I(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\left(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji} - w_{ji}^{n+1}||^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2\right)^{\frac{1}{2}}.$$

We have used above the inequality (see Corollary 4.1 in [21])

$$||Ax||_{l^2} \leq (\max_i \sum_j |a_{ij}|)||x||_{l^2}, \tag{2.21}$$

where $A = (a_{ij})_{ij}$ is a symmetric matrix. In view of Assumption 2.3 and

(2.10), we have

$$
\begin{aligned}
&(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}-w_{ji}^{n+1}||^2)^{\frac{1}{2}} \le (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}||^2)^{\frac{1}{2}} + (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2)^{\frac{1}{2}} \le \\
&C_2(||u-u^n||^2 + \sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2)^{\frac{1}{2}} + (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2)^{\frac{1}{2}} \le \\
&C_2||u-u^n|| + (1+C_2)(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2)^{\frac{1}{2}} \le \\
&C_2||u-u^{n+1}|| + (1+C_2+C_1C_2)(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \frac{\alpha}{2}||u^{n+1}-u||^2 \le \beta I(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot \\
&\Bigg[C_2||u-u^{n+1}||(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2)^{\frac{1}{2}} + \\
&(1+C_2+C_1C_2)\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2\Bigg] \le \\
&\beta I(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\Big[C_2\varepsilon||u-u^{n+1}||^2 + \\
&(1+C_2+C_1C_2+\frac{C_2}{\varepsilon})\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2\Bigg],
\end{aligned}
$$

for any $\varepsilon > 0$. With ε in (2.19), the above equation becomes,

$$
F(u^{n+1}) - F(u) \le \frac{\alpha}{2C_2\varepsilon}(1+C_2+C_1C_2+\frac{C_2}{\varepsilon})\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^2.
$$

From this equation and (2.20),

$$
F(u^{n+1}) - F(u) \le \frac{1}{C_2\varepsilon}(1+C_2+C_1C_2+\frac{C_2}{\varepsilon})\left(F(u^n) - F(u^{n+1})\right)
$$

with ε in (2.19). From the above equation, we easily get equation (2.16) with $\tilde{C}_1$ given in (2.18). Using (2.5), we see that error estimation in (2.17) can be obtained from (2.16). □

3 Multilevel Schwarz methods

We consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \ldots, J$ over the domain $\Omega \subset \mathbf{R}^d$. We write $\Omega_j = \cup_{\tau \in \mathcal{T}_{h_j}} \tau$ and we assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$ on Ω_j, $j = 1, \ldots, J-1$, and $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_J = \Omega$. Also, we assume that, if a node of $\mathcal{T}_{h_j}$ lies on $\partial\Omega_j$, then it lies on $\partial\Omega_{j+1}$, too, that is, it lies on $\partial\Omega$. Besides, we suppose that $\mathrm{dist}_{x_{j+1}\text{node of } \mathcal{T}_{h_{j+1}}}(x_{j+1}, \Omega_j) \leq Ch_j$, $j = 1, \ldots, J-1$. In this section, C denotes a generic positive constant independent of the mesh sizes, the number of meshes, as well as of the overlapping parameters and the number of subdomains in the domain decompositions which will be considered later. Since the mesh $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, we have $h_{j+1} \leq h_j$, and assume that there exists a constant γ, independent of the number of meshes or their sizes, such that

$$1 < \gamma \leq \frac{h_j}{h_{j+1}} \leq C\gamma, \quad j = 1, \ldots, J-1. \tag{3.1}$$

At each level $j = 1, \ldots, J$, we consider an overlapping decomposition $\{\Omega_j^i\}_{1 \leq i \leq I_j}$ of Ω, and assume that the mesh partition $\mathcal{T}_{h_j}$ of Ω_j supplies a mesh partition for each Ω_j^i, $1 \leq i \leq I_j$. Also, we assume that the overlapping size for the domain decomposition at the level $1 \leq j \leq J$ is δ_j. Since $h_{j+1} \leq \delta_{j+1}$, from (3.1), we have

$$\frac{h_j}{\delta_{j+1}} \leq C\gamma, \quad j = 1, \ldots, J-1. \tag{3.2}$$

In addition, we suppose that there exists a constant C such that if ω_{j+1}^i is a connected component of Ω_{j+1}^i, $j = 1, \ldots, J-1$, $i = 1, \ldots, I_j$, then

$$\mathrm{diam}(\omega_{j+1}^i) \leq Ch_j. \tag{3.3}$$

Finally, we assume that $I_1 = 1$.

At each level $j = 1, \ldots, J$, we introduce the linear finite element spaces,

$$V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, \tag{3.4}$$

and, for $i = 1, \ldots, I_j$, we write

$$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\}. \tag{3.5}$$

The functions in V_{h_j} $j = 1, \ldots, J-1$, will be extended with zero outside Ω_j and the spaces will be considered as subspaces of $H^1(\Omega)$. We denote by $||\cdot||_0$ the norm in L^2, and by $||\cdot||_1$ and $|\cdot|_1$ the norm and seminorm in

$H^1(\Omega)$, respectively. Since $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, $j = 1, \ldots, J-1$, we have

$$V_{h_1} \subset V_{h_2} \subset \ldots \subset V_{h_J}$$

We consider the two sided obstacle problem

$$u \in K : \langle F'(u), v - u \rangle \geq 0, \text{ for any } v \in K, \tag{3.6}$$

where

$$K = \{v \in V_{h_J} : \varphi \leq v \leq \psi\}, \tag{3.7}$$

with $\varphi,\ \psi \in V_{h_J}$, $\varphi \leq \psi$. We shall prove that Assumptions 2.1 and 2.3 hold for this type of convex set, and explicitly write the constant C_2 depending on the mesh and overlapping parameters. We can then conclude from Theorem 2.1 that if the functional F has the asked properties, then Algorithm 2.1 is globally convergent.

To get decompositions of the elements in V_{h_J} which satisfy the corresponding condition (2.15) in Assumption 2.3 in the linear case, Lagrange interpolation operators are used (see [24] or [26]). In the case of the variational inequalities, new nonlinear operators are needed to construct the convex sets $\mathcal{K}_j$ as in Assumption 2.1 and the element decomposition in Assumption 2.3, the so called modified interpolation operators. The utilization of such operators, even if they have not been explicitly defined, has been proposed in [18] and [19] for the complementarity problem. The operators we define below have been introduced in [3], but they have been also used in [2]. These operators allow us to analyse the two-obstacle problems, and they generalize those introduced in [20] for the one-obstacle problems.

For $j = 1, \ldots, J-1$, we define the operators $I_{h_j} : V_{h_{j+1}} \to V_{h_j}$ as follows. Let us denote by x_{ji} a node of $\mathcal{T}_{h_j}$, by ϕ_{ji} the nodal basis function associated with x_{ji} and $\mathcal{T}_{h_j}$, and by ω_{ji} the support of ϕ_{ji}. Given a $v \in V_{h_{j+1}}$, we write $I_{ji}^- v = \min_{x \in \omega_{ji}} v(x)^-$ and $I_{ji}^+ v = \min_{x \in \omega_{ji}} v(x)^+$, where $v(x)^- = \max(0, -v(x))$ and $v(x)^+ = \max(0, v(x))$. We notice that, since v is piecewise linear, $I_{ji}^- v$ or $I_{ji}^+ v$ are attained at a node of $\mathcal{T}_{h_{j+1}}$. Next, we define $I_{h_j}^- v := \sum_{x_{ji} \text{node of } \mathcal{T}_{h_j}} (I_{ji}^- v)\phi_{ji}(x)$, $I_{h_j}^+ v := \sum_{x_{ji} \text{node of } \mathcal{T}_{h_j}} (I_{ji}^+ v)\phi_{ji}(x)$, and write $I_{h_j} v = I_{h_j}^+ v - I_{h_j}^- v$. It is simple to check that if $v(x) = 0$ at a point $x \in \Omega$, then $I_{h_j} v$ vanishes in a neighborhood of x, composed by the elements τ of $\mathcal{T}_{h_j}$ containing that point. Also,

$$\begin{aligned} 0 \leq I_{h_j} v(x) \leq v(x) \text{ if } v(x) \geq 0, \\ 0 \geq I_{h_j} v(x) \geq v(x) \text{ if } v(x) \leq 0 \end{aligned} \tag{3.8}$$

at any point $x \in \Omega$. Consequently, the function

$$\theta_v(x) = \begin{cases} \frac{I_{h_j} v(x)}{v(x)} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0 \end{cases}$$

is well defined, continuous and satisfies

$$0 \le \theta_v(x) \le 1 \text{ for any } x \in \Omega. \tag{3.9}$$

Also, for any $v,\ w \in V_{h_{j+1}}$, we have

$$v \le w \text{ in } \Omega \text{ implies } I_{h_j} v \le I_{h_j} w \text{ in } \Omega. \tag{3.10}$$

We shall use these properties of the operator I_{h_j} in the following. We also recall the stability properties of the modified interpolation given in Lemma 4.2 in [3]: for any $v \in V_{h_{j+1}}$, we have

$$||I_{h_j} v - v||_0 \le C h_j C_d(h_j, h_{j+1}) |v|_1 \tag{3.11}$$

and

$$||I_{h_j} v||_0 \le ||v||_0 \text{ and } |I_{h_j} v|_1 \le C C_d(h_j, h_{j+1}) |v|_1, \tag{3.12}$$

where

$$C_d(H,h) = \begin{cases} 1 & \text{if } d = 1 \\ (\ln \frac{H}{h} + 1)^{\frac{1}{2}} & \text{if } d = 2 \\ (\frac{H}{h})^{\frac{1}{2}} & \text{if } d = 3. \end{cases} \tag{3.13}$$

It is proved in Lemma 4.2 in [3] that $||I_{h_j} v||_{0,\sigma} \le C||v||_{0,\sigma}$, but in view of (3.8), we can take $C = 1$.

Now, we define the level convex sets $\mathcal{K}_j \subset V_{h_j}$, $j = J, \ldots, 1$, satisfying Assumption 2.1. Let K be the convex set defined in (3.7), and a given $w \in K$. For the level J, we define

$$\begin{aligned} &\varphi_J = \varphi - w,\ \psi_J = \psi - w, \\ &\mathcal{K}_J = [\varphi_J, \psi_J], \ \text{ and consider a } w_J \in \mathcal{K}_J. \end{aligned} \tag{3.14}$$

At a level $j = J-1, \ldots, 1$, we define

$$\begin{aligned} &\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1}),\ \psi_j = I_{h_j}(\psi_{j+1} - w_{j+1}), \\ &\mathcal{K}_j = [\varphi_j, \psi_j], \ \text{ and consider a } w_j \in \mathcal{K}_j. \end{aligned} \tag{3.15}$$

We have

Proposition 3.1. *Assumption 2.1 holds for the convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, defined in (3.14) and (3.15), for any $w \in K$.*

Proof. Evidently, $0 \in \mathcal{K}_J$. Also, in view of (3.8), we recurrently get that $0 \in \mathcal{K}_j$ for $j = J-1, \ldots, 1$. Form the definition of $\mathcal{K}_J$, we have $w + v_J \in K$ for any $v_J \in \mathcal{K}_J$. Finally, we prove (2.6) for $j = J-1, \ldots, 1$. Let $v_j \in \mathcal{K}_j$. Using again (3.8), we get

$$\begin{aligned} &\varphi_{j+1} - w_{j+1} \le I_{h_j}(\varphi_{j+1} - w_{j+1}) = \\ &\varphi_j \le v_j \le \psi_j = I_{h_j}(\psi_{j+1} - w_{j+1}) \le \psi_{j+1} - w_{j+1}, \end{aligned}$$

ie., $w_{j+1} + v_j \in \mathcal{K}_{j+1}$. □

Now, in order to prove that Assumption 2.3 holds for the convex sets defined in (3.14) and (3.15), we consider $u,\ w \in K$ and some $w_j \in \mathcal{K}_j$, $j = J, \ldots, 1$. First, we define

$$v_J = u - w \text{ and } v_j = I_{h_j}(v_{j+1} - w_{j+1}) \text{ for } j = J-1, \ldots, 1, \tag{3.16}$$

and then,

$$\begin{aligned} &u_j = v_j - v_{j-1} = v_j - I_{h_{j-1}}(v_j - w_j) \text{ for } j = J, \ldots, 2, \\ &u_1 = v_1 = I_{h_1}(v_2 - w_2). \end{aligned} \tag{3.17}$$

With these notations, we have

Lemma 3.1. *If $\mathcal{K}_j$ are defined in (3.14) and (3.15), and v_j and u_j are defined in (3.16) and (3.17), respectively, then $v_j,\ u_j \in \mathcal{K}_j$, $j = J, \ldots, 1$, and*

$$u - w = \sum_{j=1}^{J} u_j. \tag{3.18}$$

Proof. The writing of $u-w$ as in (3.18) is evident from (3.16) and (3.17). We prove that $v_j \in \mathcal{K}_j,\ j = J, \ldots, 1$, by induction. First,

$$\varphi_J = \varphi - w \le u - w \le \psi - w = \psi_J,$$

and therefore, $v_J \in \mathcal{K}_J$. For a $j = J-1, \ldots, 1$, assuming that $v_{j+1} \in \mathcal{K}_{j+1}$, from (3.10), we have

$$\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1}) \le I_{h_j}(v_{j+1} - w_{j+1}) \le I_{h_j}(\psi_{j+1} - w_{j+1}) = \psi_j,$$

or $v_j \in \mathcal{K}_j$. For $j = J, \ldots, 2$, using (3.9), we have

$$u_j = v_j - I_{h_{j-1}}(v_j - w_j) = (1 - \theta_{v_j - w_j})v_j + \theta_{v_j - w_j} w_j,$$

and therefore, $u_j \in \mathcal{K}_j$. $\square$

The stability of the level decomposition (3.17) obtained with the modified interpolation operators is given by the previous lemma and the following result.

Lemma 3.2. *If u_j are defined in (3.17), then*

$$|u_j|_1^2 \le C(J-1)C_d(h_{j-1}, h_J)^2 [\sum_{k=2}^{J} |w_k|_1^2 + |u - w|_1^2], \tag{3.19}$$

for $j = J, \dots, 1$, *where we take* $h_0 = h_1$ *for* $j = 1$, *and*

$$(3.20) \qquad \begin{aligned} &||u_j||_0^2 \le 2||w_j||_0^2 + C(J-1)h_{j-1}^2 C_d(h_j, h_J)^2 \cdot \\ &[\sum_{k=2}^{J} |w_k|_1^2 + |u-w|_1^2], \textit{for } j = J, \dots, 2, \textit{ and} \\ &||u_1||_0^2 \le C(J-1)[||u-w||_0^2 + \sum_{j=2}^{J} ||w_j||_0^2]. \end{aligned}$$

Proof. With v_j in (3.16), we write

$$v_j - w_j = -w_j + I_{h_j}(v_{j+1} - w_{j+1}),\ j = J-1, \dots, 1,$$

and using Lemma 5.1 in [3] for $v_j - w_j$, we get

$$\begin{aligned} &|v_j|_1^2 = |I_{h_j}(v_{j+1} - w_{j+1})|_1^2 \le \\ &C(J-j)[\sum_{k=j+1}^{J-1} C_d(h_j, h_k)^2 |w_k|_1^2 + C_d(h_j, h_J)^2 |v_J - w_J|_1^2]. \end{aligned}$$

Consequently, we have

$$(3.21) \qquad |v_j|_1^2 \le C(J-j) C_d(h_j, h_J)^2 (\sum_{k=j+1}^{J} |w_k|_1^2 + |u-w|_1^2),$$

for $j = J-1, \dots, 1$. Since $u_j = v_j - v_{j-1}$, for $j = J-1, \dots, 2$, we get

$$(3.22) \qquad |u_j|_1^2 \le C(J-j+1) C_d(h_{j-1}, h_J)^2 (\sum_{k=j}^{J} |w_k|_1^2 + |u-w|_1^2).$$

Since $u_1 = v_1$, we have

$$(3.23) \qquad |u_1|_1^2 \le C(J-1) C_d(h_1, h_J)^2 (\sum_{k=2}^{J} |w_k|_1^2 + |u-w|_1^2).$$

Also, from (3.16), (3.17) and (3.12), we have

$$\begin{aligned} &|u_J|_1 = |u - w - I_{h_{J-1}}(u - w - w_J)|_1 \le \\ &(1 + CC_d(h_{J-1}, h_J))|u-w|_1 + CC_d(h_{J-1}, h_J)|w_J|_1, \end{aligned}$$

ie., we have

$$(3.24) \qquad |u_J|_1^2 \le CC_d(h_{J-1}, h_J)^2 (|w_J|_1^2 + |u-w|_1^2).$$

From (3.22), (3.23) and (3.24), we get (3.19). Now, for $j = J, \dots, 2$, from (3.11) and (3.17), we get

$$
\begin{aligned}
&||u_j||_0 \leq ||v_j - w_j - I_{h_{j-1}}(v_j - w_j)||_0 + ||w_j||_0 \leq \\
&Ch_{j-1}C_d(h_{j-1}, h_j)|v_j - w_j|_1 + ||w_j||_0 \leq \\
&Ch_{j-1}(|v_j|_1 + |w_j|_1) + ||w_j||_0,
\end{aligned}
$$

where we have used (3.1) and the definition of $C_d(H, h)$, (3.13). From this equation, we get the first equation in (3.20) for $j = J$. Also, using the above equation and (3.21), we get the first equation in (3.20) for $j = J - 1, \dots, 2$. For $j = 1$, from (3.12), we have

$$
\begin{aligned}
&||u_1||_0 = ||I_{h_1}(v_2 - w_2)||_0 \leq ||v_2 - w_2||_0 \leq \\
&||I_{h_2}(v_3 - w_3)||_0 + ||w_2||_0 \leq \dots \leq ||v_J||_0 + \sum_{j=2}^{J} ||w_j||_0,
\end{aligned}
$$

ie., the second equation in (3.20) holds. $\square$

To prove that Assumption 2.3 holds, we associate to the decomposition $\{\Omega_j^i\}_{1 \leq i \leq I_j}$ of Ω_j, some functions $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1, \cdots, I_j$, such that

$$
\begin{aligned}
&0 \leq \theta_j^i \leq 1 \text{ on } \Omega_j, \\
&\theta_j^i = 0 \text{ on } \cup_{l=i+1}^{I_j} \Omega_j^l \backslash \Omega_j^i \text{ and } \theta_j^i = 1 \text{ on } \Omega_j^i \backslash \cup_{l=i+1}^{I_j} \Omega_j^l.
\end{aligned}
\tag{3.25}
$$

Such functions θ_j^i with the above properties have been introduced in [1] and they are constructed using unity partitions of the domains $\cup_{l=i}^{I_j} \Omega_j^l$, $i = 1, \dots, I_j$, for each level $j = 1, \dots, J$. In the linear case, it suffice to consider, for each fixed level $j = 1, \dots, J$, the unity partition of Ω_j associated with its domain decomposition. For the existence of such unity partitions we can also see [22], pag. 57. Since the overlapping size of the domain decomposition on a level $j = J, \dots, 1$ is δ_j, the above functions θ_j^i can be chosen to satisfy

$$
|\partial_{x_k} \theta_j^i| \leq C/\delta_j, \text{ a.e. in } \Omega_j, \text{ for any } k = 1, \dots, d. \tag{3.26}
$$

Now, we can prove

Proposition 3.2. *Assumption 2.3 holds for the convex sets $\mathcal{K}_j$, $j = J, \dots, 1$, defined in (3.14) and (3.15). The constant C_2 is given by*

$$
C_2 = CI^2(J-1)^{\frac{1}{2}}[\sum_{j=2}^{J} C_d(h_{j-1}, h_J)^2]^{\frac{1}{2}}. \tag{3.27}
$$

Proof. Let us consider $u,\ w \in K$ and $w_{ji} \in V_{h_j}^i$ such that $w_{j1}+\ldots+w_{ji} \in \mathcal{K}_j$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$. In the construction of the convex sets $\mathcal{K}_j$, we take $w_j = \sum_{i=1}^{I_j} w_{ji}$. Then, from Lemma 3.1, there exist $u_j \in \mathcal{K}_j$, $j = J, \ldots, 1$, defined in (3.17), such that (3.18) holds. Now, for each u_j, $j = J, \ldots, 1$, we define

$$\begin{aligned} &u_{j1} = L_{h_j}(\theta_j^1 u_j + (1-\theta_j^1) w_{j1}) \text{ and} \\ &u_{ji} = L_{h_j}(\theta_j^i (u_j - \sum_{l=1}^{i-1} u_{jl}) + (1-\theta_j^i) w_{ji}), \quad i = 2, \ldots, I_j, \end{aligned} \tag{3.28}$$

with θ_j^i in (3.25), L_{h_j} being the P_1-Lagrangian interpolation. Like in Proposition 3.1 in [3] (see also [1] or [2]), where we take $v = u_j$ and $w = 0$, we can prove that

$$\begin{aligned} &u_{ji} \in V_{h_j}^i, \quad w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j,\ i = 1, \ldots, I_j, \\ &\text{and } u_j = \sum_{i=1}^{I_j} u_{ji}, \end{aligned} \tag{3.29}$$

for any $j = J, \ldots, 1$. We point out that here, the condition $w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j$ can be proved by verifying that it is satisfied only at the nodes of $\mathcal{T}_{h_j}$. From (3.18) and (3.29), we get that the first two conditions, (2.13) and (2.14), of Assumption 2.3 are satisfied.

We estimate now the constant C_2. In view of (3.28), and using (3.26) and some proprieties of the Lagrange interpolation operator (see [2] or [3]), we can write (see Proposition 3.2 in [5] for details),

$$\begin{aligned} ||u_{ji}||_1^2 \le C \Bigg(& |u_j|_1^2 + (1 + \frac{I_j - 1}{\delta_j})^2 ||u_j||_0^2 + \\ & I_j (1 + (I_j - 1)\frac{h_{j-1}}{\delta_j})^2 \sum_{k=1}^{I_j} |w_{jk}|_1^2 \Bigg). \end{aligned} \tag{3.30}$$

In view of Lemma 3.2 and (3.2), we have for $j = J, \ldots, 2$,

$$\begin{aligned} & |u_j|_1^2 + (1 + \tfrac{I_j-1}{\delta_j})^2 ||u_j||_0^2 \le C(J-1)[1 + (I_j - 1)\frac{h_{j-1}}{\delta_j}]^2 C_d(h_{j-1}, h_J)^2 \cdot \\ & [\sum_{k=2}^{J} |w_k|_1^2 + |u - w|_1^2] + C(1 + \frac{I_j - 1}{\delta_j})^2 ||w_j||_0^2 \le \\ & C(J-1) I^2 C_d(h_{j-1}, h_J)^2 [\sum_{k=2}^{J} |w_k|_1^2 + |u - w|_1^2] + C(1 + \frac{I_j - 1}{\delta_j})^2 ||w_j||_0^2. \end{aligned}$$

Consequently, from (3.30) and the above equation, we have

$$
\begin{aligned}
||u_{ji}||_1^2 \leq C \Bigg\{ & I^3 \sum_{k=1}^{I_j} |w_{jk}|_1^2 + I^2(J-1)C_d(h_{j-1}, h_J)^2 \cdot \\
& [\sum_{k=2}^{J} |w_k|_1^2 + |u-w|_1^2] + (1 + \frac{I_j - 1}{\delta_j})^2 ||w_j||_0^2 \Bigg\},
\end{aligned} \tag{3.31}
$$

for any $j = J, \ldots, 2$ and $i = 1, \ldots, I_j$. At the level $j = 1$, we do not have a domain decomposition, $I_1 = 1$, and we take $u_{11} = u_1$. In this way, from Lemma 3.2, we have

$$
||u_{11}||_1^2 \leq C(J-1)C_d(h_1, h_J)^2 (\sum_{k=2}^{J} ||w_k||_1^2 + ||u-w||_1^2). \tag{3.32}
$$

From (3.31) and (3.32), we get

$$
\begin{aligned}
\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||u_{ji}||_1^2 \leq CI^3 \Bigg\{ & I \sum_{j=2}^{J}\sum_{i=1}^{I_j} |w_{ji}|_1^2 + \\
& (J-1)\left[\sum_{j=2}^{J} C_d(h_{j-1}, h_J)^2\right]\left[\sum_{j=2}^{J} ||w_j||_1^2 + ||u-w||_1^2\right] \Bigg\} + \\
& CI \sum_{j=2}^{J} (1 + \frac{I_j - 1}{\delta_j})^2 ||w_j||_0^2.
\end{aligned} \tag{3.33}
$$

The convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, are constructed in Assumption 2.1 with $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \ldots, 1$. Consequently, using the classical Friedrchs-Poicaré inequality, (3.3) and (3.2), we have

$$
\begin{aligned}
& \sum_{j=2}^{J} [1 + \frac{I_j - 1}{\delta_j}]^2 ||w_j||_0^2 \leq I \sum_{j=2}^{J} [1 + \frac{I_j - 1}{\delta_j}]^2 \sum_{i=1}^{I_j} ||w_{ji}||_0^2 \leq \\
& CI \sum_{j=2}^{J} [1 + \frac{I_j - 1}{\delta_j}]^2 h_{j-1}^2 \sum_{i=1}^{I_j} |w_{ji}|_1^2 \leq CI^3 \sum_{j=2}^{J}\sum_{i=1}^{I_j} |w_{ji}|_1^2
\end{aligned}
$$

From this equation and (3.33), we get that the constant C_2 can be written as in (3.27).

□

As we see form the above estimations, the convergence rates given in Theorem 2.1 depend on the functional F, the maximum number of the subdomains on each level, I, and the number of levels J. The number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends in general on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rate essentially depends on the number of levels J. We now estimate the constants C_1 and C_2 as functions of J. To this end, in the remainder of this section, C will be a generic constant which does not depend on J. Writing $S_d(J) = \left[\sum_{j=2}^{J} C_d(h_{j-1}, h_J)^2\right]^{\frac{1}{2}}$ from (3.1) and (3.13), we get

$$S_d(J) = \begin{cases} (J-1)^{\frac{1}{2}} & \text{if } d = 1 \\ CJ & \text{if } d = 2 \\ C^J & \text{if } d = 3. \end{cases} \tag{3.34}$$

In this general framework, we take C_1, and β_{jk}, $j,\ k = J, \dots, 1$, as in (2.12) and (2.11) but better estimations of these constants can be given in the case of the multigrid methods in the next section. From (3.27), we get

$$C_2 = C(J-1)^{\frac{1}{2}} S_d(J), \tag{3.35}$$

Remark 3.1. 1) The results of this section have referred to problems in H^1 with Dirichlet boundary conditions, and the functions corresponding to the coarse levels have been extended with zero outside the domains Ω_j, $j = J-1, \dots, 1$. Let us assume that the problem has mixed boundary conditions: $\partial\Omega_J = \Gamma_d \cup \Gamma_n$, with Dirichlet conditions on Γ_d and Neumann conditions on Γ_n. In this case, if a node of $\mathcal{T}_{h_j}$, $j = J-1, \dots, 1$, lies in $\mathrm{Int}(\Gamma_n)$, we have to assume that all the sides of the elements $\tau \in \mathcal{T}_{h_j}$ having that node are included in Γ_n.

2) Similar convergence results with those ones presented in this section can be obtained for problems in $(H^1)^d$.

4 Multigrid methods

In the above multilevel methods a mesh is the refinement of that on the previous level, but the domain decompositions are almost independent from one level to another. We obtain similar multigrid methods by decomposing the level domains by the supports of the nodal basis functions on that level. Consequently, the subspaces $V_{h_j}^i$, $i = 1, \dots, I_j$, are one-dimensional spaces spanned by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \dots, 1$. As we already said, in this case, we prove that the constant C_1 in (2.12) and the sums $\sum_{k=1}^{J} \beta_{jk}$, $j = J, \dots, 1$, with β_{jk} in (2.11), can be expressed independently of the number J of levels. We point out that multigrid Algorithm 2.1 represents a classical V-cycle iteration. Evidently, similar results can be given for the W-cycle multigrid iterations. At the end of this section, we write the convergence rate of the algorithm depending on the number of the levels.

The proof of the inequalities in Assumption 2.2 are closed related with the strengthened Cauchy-Schwarz inequality (see [24] and [26], for instance). The proof of (2.9) can be found in [21] and it essentially stands on the simple inequalities

$$(4.1)\qquad ||v_{ji}||_{0,supp(v_{kl})} \leq C(\tfrac{h_k}{h_j})^{\frac{d}{2}}||v_{ji}||_0,\ |v_{ji}|_{1,supp(v_{kl})} \leq C(\tfrac{h_k}{h_j})^{\frac{d}{2}}|v_{ji}|_1,$$

for any $v_{ji} \in V_{h_j}^i$, $v_{kl} \in V_{h_k}^l$ with $j \leq k$, $j,k = J,\dots,1$, $i = 1,\dots,I_j$ and $l = 1,\dots,I_k$. Writing $\gamma_{kj} = \frac{1}{\gamma^{|k-j|\frac{d}{2}}}$, in view of (3.1) and (4.1), we get

$$(4.2)\qquad ||v_{ji}||_{0,supp(v_{kl})} \leq C\gamma_{kj}||v_{ji}||_0,\ |v_{ji}|_{1,supp(v_{kl})} \leq C\gamma_{kj}|v_{ji}|_1,$$

where C is independent of the meshes or their number. In this way, we get that (2.9) holds for $\beta_{kj} = C\gamma_{kj},\ k,j = J,\dots,1$, and

$$(4.3)\qquad \sum_{j=1}^{J}\beta_{kj} = C\sum_{j=1}^{J}\gamma_{kj} \leq C\frac{\gamma^{\frac{d}{2}}}{\gamma^{\frac{d}{2}}-1},$$

for any $k = J,\dots,1$.

Also, there are well-known proofs for inequality (2.10). The proof we give in the following can be easily generalized for the norm in $W^{1,s}$, $1 < s < \infty$ (see [5]).

Lemma 4.1. *Constant C_1 in (2.10) can be estimated as $C_1 = \left(2CI\frac{\gamma^{\frac{d}{2}}}{\gamma^{\frac{d}{2}}-1}\right)^{\frac{1}{2}}$, where C is the constant in (4.2).*

Proof. We prove the lemma for the norm $||\cdot||_0$. The proof for the derivatives in the seminorm $|\cdot|_1$ is identical. From (4.2), we get

$$\int_\Omega v_{ji}v_{kl} \leq C\gamma_{kj}||v_{ji}||_0||v_{kl}||_0$$

for any $v_{ji} \in V_{h_j}^i$, $v_{kl} \in V_{h_k}^l$ with $j \leq k$, $j,k = J,\dots,1$, $i = 1,\dots,I_j$ and $l = 1,\dots,I_k$. Similar inequalities can be written for the derivatives of

v_{ji} and v_{kl}. In view of this inequality, (2.21) and (4.3), we get

$$||\sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}||_0^2 \leq \int_\Omega \left(\sum_{j=1}^{J}\sum_{i=1}^{I_j}|w_{ji}|\right)^2 =$$

$$\sum_{k_2=1}^{J}\sum_{i_2=1}^{I_{k_2}}\sum_{k_1=1}^{J}\sum_{i_1=1}^{I_{k_1}}\int_\Omega |w_{k_2i_2}||w_{k_1i_1}| \leq$$

$$2\sum_{j_2=1}^{J}\sum_{i_2=1}^{I_{j_2}}\sum_{j_1=j_2}^{J}\sum_{i_1=1}^{I_{j_1}}\int_\Omega |w_{j_2i_2}||w_{j_1i_1}| \leq$$

$$2C\sum_{j_2=1}^{J}\sum_{i_2=1}^{I_{j_2}}\sum_{j_1=j_2}^{J}\sum_{i_1=1}^{I_{j_1}}\left(\gamma_{j_2j_1}(\int_\Omega |w_{j_2i_2}|^2)^{\frac{1}{2}}(\int_\Omega |w_{j_1i_1}|^2)^{\frac{1}{2}}\right) \leq$$

$$2C\sum_{j_2=1}^{J}\left(\sum_{i_2=1}^{I_{j_2}}(\int_\Omega |w_{j_2i_2}|^2)^{\frac{1}{2}}\right)\left(\sum_{j_1=1}^{J}\gamma_{j_2j_1}\sum_{i_1=1}^{I_{j_1}}(\int_\Omega |w_{j_1i_1}|^2)^{\frac{1}{2}}\right)$$

$$2C\left[\sum_{j_2=1}^{J}\left(\sum_{i_2=1}^{I_{j_2}}(\int_\Omega |w_{j_2i_2}|^2)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}\cdot$$

$$\left[\sum_{j_2=1}^{J}\left(\sum_{j_1=1}^{J}\gamma_{j_2j_1}\sum_{i_1=1}^{I_{j_1}}(\int_\Omega |w_{j_1i_1}|^2)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \leq$$

$$2C\max_{j_2}\sum_{j_1=1}^{J}\gamma_{j_2j_1}\left[\sum_{j_2=1}^{J}\left(\sum_{i_2=1}^{I_{j_2}}(\int_\Omega |w_{j_2i_2}|^2)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}\cdot$$

$$\left[\sum_{j_1=1}^{J}\left(\sum_{i_1=1}^{I_{j_1}}(\int_\Omega |w_{j_1i_1}|^2)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \leq 2CI\frac{\gamma^{\frac{d}{2}}}{\gamma^{\frac{d}{2}}-1}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||_0^2.$$

□

From the above proofs, we can conclude that, in the case of the multigrid methods, we can consider C_1 and $\max_{k=J,\ldots,1}\sum_{j=1}^{J}\beta_{kj}$ as some constants independent of J and mesh parameters. Using the estimations of C_2 in (3.35), $\tilde{C}_1$ in (2.18), and the error estimation of Algorithm 2.1 in Theorem 2.1, we have

Corollary 4.1. *As a function of the number J of levels, the error estimate of the multigrid method given by Algorithm 2.1 can be written as*

$$(4.4) \qquad ||u^n - u||_1^2 \leq \tilde{C}_0\left(1 - \frac{1}{1+\tilde{C}_1(J)}\right)^n,\ \tilde{C}_1(J) = CJS_d(J)^2$$

where $S_d(J)$ is defined in (3.34), and $\tilde{C}_0$ is a constant independent of J.

We make now some remarks on the above error estimations. Algorithm 2.1 is a standard monotone multigrid method in the sense of Kornhuber (see [13] and [10]), whose level obstacles become those proposed by Mandel in [18] in the case of the complementarity problems. Our analysis refer to the two sided obstacle problems and the above convergence results give a global rate estimation. For $d = 3$, it is well known that the convergence rate deteriorates exponentially by increasing J, and it is confirmed by our error estimate (4.4). In the case $d = 2$, we can compare the convergence rates we have obtained for Algorithm 2.1 with similar ones in the literature. In this case, from (4.4), we get that the global convergence of Algorithm 2.1 is $1-\frac{1}{1+CJ^3}$. The same estimate, of $1-\frac{1}{1+CJ^3}$, is obtained in [13] for the asymptotic convergence rate of the standard monotone multigrid methods for the complementarity problem. In [10], it is mentioned that, for this method, the asymptotic rate may be of $1-\frac{1}{1+CJ^2}$, or even of $1-\frac{1}{1+CJ}$, under some conditions. The numerical experiments given in [13] and [10] also confirm our theoretical results. For the numerical examples given there, the standard monotone multigrid method has an almost uniform convergence rate during the iteration, i.e. its global convergence rate coincides with the asymptotic one.

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