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Relaxation procedures for an iterative MFS algorithm for two-dimensional steady-state isotropic heat conduction Cauchy problems

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ABSTRACT

We investigate two algorithms involving the relaxation of either the given Dirichlet data (boundary temperatures) or the prescribed Neumann data (normal heat fluxes) on the over-specified boundary in the case of the alternating iterative algorithm of Kozlov et al. [26] applied to two-dimensional steady-state heat conduction Cauchy problems, i.e. Cauchy problems for the Laplace equation. The two mixed, well-posed and direct problems corresponding to each iteration of the numerical procedure are solved using a meshless method, namely the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion. The iterative MFS algorithms with relaxation are tested for Cauchy problems associated with the Laplace operator in various two-dimensional geometries to confirm the numerical convergence, stability, accuracy and computational efficiency of the method.

1. Introduction

A classical and quite often encountered inverse problem in heat transfer is the so-called Cauchy problem. For such a problem, the boundary of the solution domain, the thermal conductivities and/or the heat sources are all known, while the boundary conditions are incomplete. More precisely, both Dirichlet (temperature) and Neumann (normal heat flux) conditions are prescribed on a part of the boundary, while on the remaining portion of the boundary no data are available. It is well known that Cauchy problems are generally ill-posed, see e.g. Hadamard [17], in the sense that the existence, uniqueness and stability of their solutions are not always guaranteed. Consequently, a special treatment of these problems is required.

There are numerous important contributions in the literature, as well as various approaches, to the theoretical and numerical solutions of the Cauchy problem associated with the steady-state heat conduction in isotropic media, i.e. the Laplace equation. The method of quasi-reversibility, in conjunction with a finite-difference method (FDM) and Carleman-type estimates, were employed by Klibanov and Santosa [25] to solve this inverse problem. Kozlov et al. [26] proposed an alternating iterative algorithm for the stable solution of this problem, which was implemented using the boundary element method (BEM) by Lesnic et al. [28]. Ang et al. [2] reformulated the Cauchy problem as an integral equation problem and solved the latter by using the Fourier transform, together with the Tikhonov regularization method. Reinhardt et al. [47] proved that the standard five-point FDM approximation to the Cauchy problem for the Laplace equation satisfies some stability estimates and hence it turns out to be a well-posed problem, provided that a certain bounding requirement is fulfilled. As a result of a variational approach to the Cauchy problem, the conjugate gradient method, in conjunction with the BEM, was proposed by Hào and Lesnic [19] in order to obtain a stable solution. Cheng et al. [6] transformed the original problem into a moment problem by using Green's formula and they also provided an error estimate for the numerical solution. Hon and Wei [21] converted the Cauchy problem into a classical moment problem whose numerical approximation can be achieved and also provided a convergence proof based on Backus-Gilbert algorithm. Cimetière et al. [8] reduced the Cauchy problem for the Laplace equation to solving a sequence of optimization problems under equality constraints using the finite element method (FEM). The minimization functional consists of two terms that measure the gap between the optimal element and the over-specified data and the gap between the optimal element and the previous optimal element (regularization term), respectively. This method was later implemented using the BEM by Delvare et al. [11]. Cimetière et al. [9] reduced the solution of harmonic Cauchy problems to the resolution of a fixed point process, while the authors implemented numerically the proposed method by employing both the BEM and the FEM. Jourhmane et al. [24] developed three relaxation procedures in order to increase the rate of convergence of the algorithm of

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Kozlov et al. [26], at the same time selection criteria for the variable relaxation factors having been provided. Bourgeois [3] approached the Cauchy problem for the Laplace equation by the mixed formulation of the method of quasi-reversibility, which finally led to a $\mathcal{C}^0$ FEM. Andrieux et al. [1] introduced an energy-like error functional and converted the inverse problem into an optimization problem. In order to improve the reconstruction of the normal derivatives, Delvare and Cimetière [10] extended the method originally proposed by Cimetière et al. [8] to a higher-order one, which was implemented using the BEM. On assuming the available data to have a Fourier expansion, Liu [31] applied a modified indirect Trefftz method to solve the Cauchy problem for the Laplace equation.

The method of fundamental solutions (MFS) is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of solutions to linear partial differential equations. Like the BEM, the MFS is applicable when a fundamental solution of the governing PDE is explicitly known. Since its introduction as a numerical method in the late 1970s by Mathon and Johnston [42], it has been successfully applied to a large variety of physical problems, an account of which may be found in the excellent survey papers by Cho et al. [7], Fairweather and Karageorghis [13], Fairweather et al. [14] and Golberg and Chen [15]. The ease of implementation of the MFS and its low computational cost make it an ideal candidate for inverse problems as well. For these reasons, the MFS, in conjunction with various regularization methods (e.g. the Tikhonov regularization method, Morozov's discrepancy principle, singular value decomposition), have been used increasingly over the last decade for the numerical solution of inverse problems. For example, the Cauchy problem associated with the heat conduction equation [12,29,35,36,43,48,51–54], linear elasticity [32,39], steady-state heat conduction in functionally graded materials [33], Helmholtz-type equations [23,34,37,40], Stokes problems [5], the biharmonic equation [41] etc. have been successfully addressed by using the MFS.

Recently, Marin [36] solved numerically the Cauchy problem in steady-state isotropic heat conduction (Laplace equation) by applying, in an iterative manner, the MFS for the alternating iterative algorithm of Kozlov et al. [26]. At each iteration, Marin [36] solved two mixed, well-posed and direct problems using the MFS, in conjunction with the Tikhonov regularization method. For each of the aforementioned direct problems, the optimal value of the regularization parameter was chosen according to the generalized cross-validation (GCV) criterion. Consequently, an iterative procedure, which provides the selection of the optimal regularization parameter, occurs within each step of the iterative algorithm of Kozlov et al. [26] and hence the computational cost of the iterative MFS-based algorithm is increased. In order to overcome this inconvenience, we decided to employ two relaxation procedures, as proposed by Jourhmane et al. [24], for the iterative MFS-based algorithm implemented by Marin [36] and study the influence of the relaxation parameter upon the rate of convergence of the modified method. The efficiency of these relaxation procedures is tested for Cauchy problems associated with the two-dimensional Laplace operator in simply and doubly connected, convex and concave domains, with smooth or piecewise smooth boundaries.

2. Mathematical formulation

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, where d is the dimension of the space where the problem is posed, usually $d \in \{1,2,3\}$, occupied by an isotropic medium. We assume that Ω is bounded by a piecewise smooth curve $\partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $H^1(\Omega)$ be the Sobolev space of real-valued functions in Ω endowed with the standard norm, see e.g. Lions and Magenes [30]. We denote by $H_0^1(\Omega)$ and $H_{\Gamma_i}^1(\Omega)$, $i=1, 2$, the subspaces of functions from $H^1(\Omega)$ that vanish on $\partial\Omega$ and Γ_i, $i=1, 2$, respectively.

The space of traces of functions from $H^1(\Omega)$ to $\partial\Omega$ is denoted by $H^{1/2}(\partial\Omega)$, while the restrictions of the functions belonging to the space $H^{1/2}(\partial\Omega)$ to the subset $\Gamma_i \subset \partial\Omega$, $i=1,2$, define the space $H^{1/2}(\Gamma_i)$, $i=1, 2$. The set of real valued functions in $\partial\Omega$ with compact support in Γ_i, $i=1, 2$, and bounded first-order derivatives are dense in $H^{1/2}(\Gamma_i)$, $i=1, 2$. Furthermore, we also define the space $H_{00}^{1/2}(\Gamma_i)$, $i=1, 2$, that consists of functions from $H^{1/2}(\partial\Omega)$ and vanishing on Γ_{3-i}, $i=1, 2$. The space $H_{00}^{1/2}(\Gamma_i)$, $i=1, 2$, is a subspace of $H^{1/2}(\partial\Omega)$ with the norm given by

$$\|f\|_{H_{00}^{1/2}(\Gamma_i)} = \left(\int_{\Gamma_i} \frac{f^2(\mathbf{x})}{\mathrm{dist}(\mathbf{x},\Gamma_i)} \mathrm{d}\Gamma(\mathbf{x}) + \int_{\Gamma_i}\int_{\Gamma_i} \frac{|f(\mathbf{x}) - f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^d} \mathrm{d}\Gamma(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{y}) \right)^{1/2}. \tag{1}$$

It should be mentioned that the space of restrictions from $H_{00}^{1/2}(\Gamma_i)$ to Γ_i, $i=1, 2$, is dense in $H^{1/2}(\Gamma_i)$, $i=1, 2$. Nonetheless, $H_{00}^{1/2}(\Gamma_i) \neq H^{1/2}(\Gamma_i)$. Finally, we denote by $(H_{00}^{1/2}(\Gamma_i))^*$ the dual space of $H_{00}^{1/2}(\Gamma_i)$, $i=1, 2$.

In this paper, we refer to steady-state heat conduction applications in isotropic homogeneous media in the absence of heat sources. Consequently, the function $u(\mathbf{x})$ denotes the temperature at a point $\mathbf{x} \in \Omega$ and satisfies the heat balance equation

$$\nabla^2 u(\mathbf{x}) \equiv \sum_{i=1}^{d} \partial_i \partial_i u(\mathbf{x}) = 0, \quad \mathbf{x} = (x_1, \ldots, x_d) \in \Omega, \tag{2}$$

where $\partial_i \equiv \partial/\partial x_i$. We now let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector at $\partial\Omega$ and $q(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$ defined by

$$q(\mathbf{x}) \equiv -\nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = -\sum_{i=1}^{d} \partial_i u(\mathbf{x}) \mathrm{n}_i(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{3}$$

In the direct problem formulation, the knowledge of the location, shape and size of the entire boundary $\partial\Omega$, the temperature and/or normal heat flux on the entire boundary $\partial\Omega$ gives the corresponding Dirichlet, Neumann, Robin, or mixed boundary conditions which enable us to determine the unknown boundary conditions, as well as the temperature distribution in the solution domain. In many engineering problems, a different and more interesting situation occurs when both the temperature and the normal heat flux are prescribed on a part of the boundary, say Γ_1, while no boundary conditions are supplied on the remaining part of the boundary $\Gamma_2 = \partial\Omega \backslash \Gamma_1$. More precisely, we consider the following *Cauchy problem* for steady-state heat conduction in an isotropic homogeneous medium:

$$\nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{4a}$$

$$u(\mathbf{x}) = \tilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{4b}$$

$$q(\mathbf{x}) = \tilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{4c}$$

where $\tilde{u} \in H^{1/2}(\Gamma_1)$ and $\tilde{q} \in (H_{00}^{1/2}(\Gamma_1))^*$ are prescribed Dirichlet and Neumann boundary conditions, respectively.

A necessary condition for the Cauchy problem given by Eqs. (4a)–(4c) to be identifiable is that $\mathrm{meas}(\Gamma_1) > 0$, see Isakov [22]. However, in the discretised version of the aforementioned Cauchy problem, the corresponding identifiability condition reduces to $\mathrm{meas}(\Gamma_1) \geq \mathrm{meas}(\Gamma_2)$, see, e.g. Lesnic et al. [28]. This inverse problem is much more difficult to solve both analytically and numerically than the direct problem, since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well known that this solution is unstable with respect to small perturbations into the data on Γ_1, see Hadamard [17]. Thus the problem is ill-posed and, therefore, regularization methods are required in order to solve accurately the inverse problem (4a)–(4c) for the Laplace equation.

3. Alternating iterative algorithms with relaxation

In this section, we present two alternating iterative algorithms with relaxation, as proposed by Jourhmane et al. [24], which aim to reduce the computational time of the alternating iterative algorithm introduced by Kozlov et al. [26] for the simultaneous and stable reconstruction of both the unknown temperature $u|_{\Gamma_2}$ and the normal heat flux $q|_{\Gamma_2}$.

Alternating iterative algorithm with relaxation I:

Step 1: (i) If $k=1$ then specify an initial guess for the normal heat flux on Γ_2, namely $q^{(2k-1)} \in (H_{00}^{1/2}(\Gamma_2))^*$.

(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$\nabla^2 u^{(2k-1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{5a}$$

$$q^{(2k-1)}(\mathbf{x}) = \tilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{5b}$$

$$u^{(2k-1)}(\mathbf{x}) = u^{(2k-2)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2 \tag{5c}$$

to determine $u^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $q^{(2k-1)}(\mathbf{x}) \equiv -\nabla u^{(2k-1)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 2: Update the unknown Neumann data on Γ_2 by setting

$$\xi^{(k)}(\mathbf{x}) = \begin{cases} q^{(2k-1)}(\mathbf{x}) & \text{for } k=1, \\ \omega\, q^{(2k-1)}(\mathbf{x}) + (1-\omega)\xi^{(k-1)}(\mathbf{x}) & \text{for } k \geq 2, \end{cases} \quad \mathbf{x} \in \Gamma_2, \tag{6}$$

where $\omega \in (0,2)$ is a fixed relaxation factor.

Having constructed the approximation $u^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$\nabla^2 u^{(2k)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{7a}$$

$$u^{(2k)}(\mathbf{x}) = \tilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{7b}$$

$$q^{(2k)}(\mathbf{x}) = \xi^{(k)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2 \tag{7c}$$

is solved in order to determine $u^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $u^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 3: Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 3.1. The value $\omega = 1$ in Eq. (6) corresponds to the alternating iterative algorithm introduced by Kozlov et al. [26] with an initial guess for the Neumann data, while the values $\omega \in (0,1)$ and $\omega \in (1,2)$ in Eq. (6) correspond to the alternating iterative algorithm introduced by Kozlov et al. [26] with an initial guess for the Neumann data and a constant under- and over-relaxation factor, respectively.

Alternating iterative algorithm with relaxation II:

Step 1: (i) If $k=1$ then specify an initial guess for the boundary temperature on Γ_2, namely $u^{(2k-1)} \in H^{1/2}(\Gamma_2)$.

(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$\nabla^2 u^{(2k-1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{8a}$$

$$u^{(2k-1)}(\mathbf{x}) = \tilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{8b}$$

$$q^{(2k-1)}(\mathbf{x}) = q^{(2k-2)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2 \tag{8c}$$

to determine $u^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $u^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 2: Update the unknown Dirichlet data on Γ_2 by setting

$$\eta^{(k)}(\mathbf{x}) = \begin{cases} u^{(2k-1)}(\mathbf{x}) & \text{for } k=1, \\ \omega u^{(2k-1)}(\mathbf{x}) + (1-\omega)\, \eta^{(k-1)}(\mathbf{x}) & \text{for } k \geq 2, \end{cases} \quad \mathbf{x} \in \Gamma_2, \tag{9}$$

where $\omega \in (0,2)$ is a fixed relaxation factor.

Having constructed the approximation $u^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$\nabla^2 u^{(2k)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{10a}$$

$$q^{(2k)}(\mathbf{x}) = \tilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{10b}$$

$$u^{(2k)}(\mathbf{x}) = \eta^{(k)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2 \tag{10c}$$

is solved in order to determine $u^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $q^{(2k)}(\mathbf{x}) \equiv -\nabla u^{(2k)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 3: Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 3.2. The value $\omega = 1$ in Eq. (9) corresponds to the alternating iterative algorithm introduced by Kozlov et al. [26] with an initial guess for the Dirichlet data, while the values $\omega \in (0,1)$ and $\omega \in (1,2)$ in Eq. (9) correspond to the alternating iterative algorithm introduced by Kozlov et al. [26] with an initial guess for the Dirichlet data and a constant under- and over-relaxation factor, respectively.

The convergence of the alternating iterative algorithm with relaxation II presented above can be recast in the following convergence theorem, with the mention that a similar result can also be obtained for the alternating iterative algorithm with relaxation I:

Theorem 3.1. *Let $\tilde{u} \in H^{1/2}(\Gamma_1)$ and $\tilde{q} \in (H_{00}^{1/2}(\Gamma_1))^*$, and assume that the Cauchy problem* (4a)–(4c) *has a solution $u \in H^1(\Omega)$. Let $u^{(k)}$ be the k-th approximate solution in the alternating procedure* II *described above. Then there exists a number $1 < b \leq 2$ such that when the relaxation parameter ω is chosen with $1 \leq \omega \leq b$, then*

$$\lim_{k \to \infty} \| u - u^{(k)} \|_{H^1(\Omega)} = 0 \tag{11}$$

for any initial data element $\eta^{(1)} \in H^{1/2}(\Gamma_2)$.

The proof for this theorem in the case of the proposed relaxation algorithms associated with the Cauchy problem for the Laplace equation is similar to that for the corresponding relaxation algorithms for the Cauchy problem in elasticity, see Marin and Johansson [38]. The proof given by Marin and Johansson [38] is based on the reformulation of the Cauchy problem (4a)–(4c) as a fixed point operator equation with a self-adjoint, injective, positive definite and non-expansive operator, while the scheme is shown to be a fixed point iteration for that equation. An alternative proof for the convergence result can also be found in Jourhmane et al. [24]. As reported by Marin and Johansson [38] for Cauchy problems associated with the Navier-Lamé system of elasticity, it was also found for two-dimensional steady-state isotropic heat conduction Cauchy problems that a relaxation factor $\omega > 2$ cannot be employed since the iterative process becomes divergent in such a situation.

It is important to mention that, in general, the iterative algorithms described above do not converge if the mixed problems in the steps 1 and 2 of the algorithms are replaced by Dirichlet or Neumann problems. In addition, the Neumann direct problem associated with the Laplace equation is ill-posed owing to the non-uniqueness or non-existence of solution with respect to whether the integral of the normal heat flux q over the boundary $\partial\Omega$ vanishes or not, respectively.

4. Method of fundamental solutions

The fundamental solution G of the heat balance equation (2) or (4a) for two-dimensional steady-state heat conduction in isotropic homogeneous media, i.e. the Laplace equation, is given by [13]

$$G(\mathbf{x},\xi)=\frac{1}{2\pi}\log\frac{1}{\|\mathbf{x}-\xi\|},\quad \mathbf{x}\in\overline{\Omega},\quad \xi\in\mathbb{R}^2\setminus\overline{\Omega}, \tag{12}$$

where ξ is a singularity or source point. The main idea of the MFS consists of approximating the temperature in the solution domain by a linear combination of fundamental solutions with respect to M singularities $\xi^{(j)}$, $j=1,\ldots,M$, in the form

$$u(\mathbf{x})\approx u_M(\mathbf{c},\xi;\mathbf{x})=\sum_{j=1}^{M}c_jG(\mathbf{x},\xi^{(j)}),\quad \mathbf{x}\in\overline{\Omega}, \tag{13}$$

where $\mathbf{c}=[c_1,\ldots,c_M]^{\mathsf{T}}$ and $\xi\in\mathbb{R}^{2M}$ is a vector containing the coordinates of the singularities $\xi^{(j)}$, $j=1,\ \ldots,\ M$. On taking into account the definitions of the normal heat flux (3) and the fundamental solution for the two-dimensional Laplace equation (12) then the normal heat flux, through a curve defined by the outward unit normal vector $\mathbf{n}(\mathbf{x})$, can be approximated on the boundary $\partial\Omega$ by

$$q(\mathbf{x})\approx q_M(\mathbf{c},\xi;\mathbf{x})=\sum_{j=1}^{M}c_jH(\mathbf{x},\xi^{(j)}),\quad \mathbf{x}\in\partial\Omega, \tag{14}$$

where

$$H(\mathbf{x},\xi)=-\nabla_{\mathbf{x}}G(\mathbf{x},\xi)\cdot\mathbf{n}(\mathbf{x})=\frac{1}{2\pi}\frac{(\mathbf{x}-\xi)\cdot\mathbf{n}(\mathbf{x})}{\|\mathbf{x}-\xi\|^2},\quad \mathbf{x}\in\partial\Omega,\ \xi\in\mathbb{R}^2\setminus\overline{\Omega}. \tag{15}$$

Next, we select the N_1 MFS collocation points $\{\mathbf{x}^{(i)}\}_{i=1}^{N_1}$ on the boundary Γ_1 and the N_2 MFS collocation points $\{\mathbf{x}^{(i)}\}_{i=N_1+1}^{N_1+N_2}$ on the boundary Γ_2, such that the total number of MFS collocation points used to discretize the boundary $\partial\Omega$ of the solution domain Ω is given by $N = N_1 + N_2$.

According to the MFS approximations (13) and (14), the discretized versions of the boundary value problems (5a)–(5c) and (7a)–(7c), or (8a)–(8c) and (10a) and (10c) recast as

$$\mathbf{A}^{(1)}\mathbf{c}^{(2k-1)}=\mathbf{b}^{(2k-1)},\quad k>1, \tag{16}$$

and

$$\mathbf{A}^{(2)}\mathbf{c}^{(2k)}=\mathbf{b}^{(2k)},\quad k\geq 1, \tag{17}$$

respectively. For example, in the case of the alternating iterative algorithm with relaxation I, the components of the MFS matrices and right-hand side vectors corresponding to Eqs. (16) and (17) are given by

$$A_{ij}^{(1)}=\begin{cases}H(\mathbf{x}^{(i)},\xi^{(j)}), & i=1,\ldots,N_1,\quad j=1,\ldots,M,\\ G(\mathbf{x}^{(i)},\xi^{(j)}), & i=N_1+1,\ldots,N_1+N_2,\quad j=1,\ldots,M,\end{cases} \tag{18a}$$

$$b_i^{(2k-1)}=\begin{cases}\tilde{q}(\mathbf{x}^{(i)}), & i=1,\ldots,N_1,\\ u^{(2k-2)}(\mathbf{x}^{(i)}), & i=N_1+1,\ldots,N_1+N_2,\end{cases} \tag{18b}$$

and

$$A_{ij}^{(2)}=\begin{cases}G(\mathbf{x}^{(i)},\xi^{(j)}), & i=1,\ldots,N_1,\quad j=1,\ldots,M,\\ H(\mathbf{x}^{(i)},\xi^{(j)}), & i=N_1+1,\ldots,N_1+N_2,\quad j=1,\ldots,M,\end{cases} \tag{19a}$$

$$b_i^{(2k)}=\begin{cases}\tilde{u}(\mathbf{x}^{(i)}), & i=1,\ldots,N_1,\\ \xi^{(k)}(\mathbf{x}^{(i)}), & i=N_1+1,\ldots,N_1+N_2,\end{cases} \tag{19b}$$

respectively. Similar expressions are obtained in the case of the alternating iterative algorithm with relaxation II, and therefore they are not presented.

Each of Eqs. (16) and (17) represents a system of N linear algebraic equations with M unknowns, namely the MFS coefficients $\mathbf{c}^{(2k-1)}=[c_1^{(2k-1)},\ldots,c_M^{(2k-1)}]^{\mathsf{T}}$ and $\mathbf{c}^{(2k)}=[c_1^{(2k)},\ldots,c_M^{(2k)}]^{\mathsf{T}}$, respectively. It should be noted that in order to uniquely determine the solutions $\mathbf{c}^{(2k-1)}\in\mathbb{R}^M$ and $\mathbf{c}^{(2k)}\in\mathbb{R}^M$ to the systems of linear algebraic equations (16) and (17), respectively, the number N of MFS boundary collocation points on the boundary $\partial\Omega$ and the number M of singularities must satisfy the inequality $M\leq N$. However, the systems of linear algebraic equations (16) and (17) cannot be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution in the case of noisy Cauchy data on Γ_1.

In order to implement the MFS, the location of the singularities has to be determined and this is usually achieved by considering either the static or the dynamic approach. In the static approach, the singularities are pre-assigned and kept fixed throughout the solution process, whilst in the dynamic approach, the singularities and the unknown coefficients are determined simultaneously during the solution process, see Fairweather and Karageorghis [13]. Thus the dynamic approach transforms the inverse problem into a more difficult nonlinear ill-posed problem which is also computationally much more expensive. The advantages and disadvantages of the MFS with respect to the location of the fictitious sources are described at length in Heise [20] and Burgess and Maharejin [4]. Recently, Gorzelańczyk and Kołodziej [16] thoroughly investigated the performance of the MFS with respect to the shape of the pseudo-boundary on which the source points are situated, proving that, for the same number of boundary collocation points and sources, more accurate results are obtained if the shape of the pseudo-boundary is similar to that of the boundary of the solution domain. Therefore, we have decided to employ the static approach in our computations, at the same time accounting for the findings of Gorzelańczyk and Kołodziej [16].

5. Regularization

Since the right-hand sides of the systems of linear algebraic equations (16) and (17) are in general polluted by noise, the retrieval of accurate and stable solutions to Eqs. (16) and (17) is very important for obtaining physically meaningful numerical results. For perturbed right-hand sides in Eqs. (16) and (17), the direct inversion of these equations or, equivalently, a least-squares minimization applied to Eqs. (16) and (17) will fail to produce stable, accurate and physically meaningful numerical solutions. It is the purpose of this section to present a classical regularization procedure for obtaining such solutions to the systems of linear algebraic equations (16) and (17), as well as details regarding the optimal choice of the regularization parameter.

5.1. Tikhonov regularization method

Several regularization techniques used for the stable solution of systems of linear and nonlinear algebraic equations are available in the literature, such as the singular value decomposition [18], the Tikhonov regularization method [49] and various iterative methods [27]. In this study, we have decided to employ the Tikhonov regularization method.

Consider the following system of linear algebraic equations:

$$\mathbf{Ac}=\mathbf{b}, \tag{20}$$

where $N \geq M$, $\mathbf{A} \in \mathbb{R}^{N \times M}$, $\mathbf{c} \in \mathbb{R}^M$ and $\mathbf{b} \in \mathbb{R}^N$. Note that Eq. (20) may describe each of the MFS systems of linear equations (16) and (17), provided that

$$\mathbf{A} = \mathbf{A}^{(1)}, \quad \mathbf{c} = \mathbf{c}^{(2k-1)}, \quad \mathbf{b} = \mathbf{b}^{(2k-1)}, \ k > 1, \tag{21}$$

and

$$\mathbf{A} = \mathbf{A}^{(2)}, \quad \mathbf{c} = \mathbf{c}^{(2k)}, \quad \mathbf{b} = \mathbf{b}^{(2k)}, \ k \geq 1, \tag{22}$$

respectively. The Tikhonov zeroth-order regularized solution to the generically written system of linear algebraic equations (20) is sought as, see Tikhonov and Arsenin [49]

$$\mathbf{c}_\lambda \in \mathbb{R}^M : \quad \mathcal{F}_\lambda(\mathbf{c}_\lambda) = \min_{\mathbf{c} \in \mathbb{R}^M} \mathcal{F}_\lambda(\mathbf{c}), \tag{23}$$

where $\mathcal{F}_\lambda$ represents the Tikhonov zeroth-order regularization functional given by, see Tikhonov and Arsenin [49]

$$\mathcal{F}_\lambda(\cdot) : \mathbb{R}^M \longrightarrow [0,\infty), \quad \mathcal{F}_\lambda(\mathbf{c}) = \|\mathbf{Ac} - \mathbf{b}\|^2 + \lambda^2 \|\mathbf{c}\|^2, \tag{24}$$

and $\lambda > 0$ is the regularization parameter to be prescribed. Formally, the Tikhonov regularized solution $\mathbf{c}_\lambda$ of the problem (20) is given as the solution of the normal equation

$$(\mathbf{A}^{\mathsf{T}}\mathbf{A} + \lambda^2 \mathbf{I}_M)\mathbf{c} = \mathbf{A}^{\mathsf{T}}\mathbf{b}, \tag{25}$$

where $\mathbf{I}_M \in \mathbb{R}^{M \times M}$ is the identity matrix, namely

$$\mathbf{c}_\lambda = \mathbf{A}^\dagger \mathbf{b}, \quad \mathbf{A}^\dagger \equiv (\mathbf{A}^{\mathsf{T}}\mathbf{A} + \lambda^2 \mathbf{I}_M)^{-1}\mathbf{A}^{\mathsf{T}}. \tag{26}$$

To summarize, the Tikhonov regularization method solves a constrained minimization problem using a smoothness norm in order to provide a stable solution which fits the data and also has a minimum structure.

5.2. Selection of the optimal regularization parameter

The performance of regularization methods depends crucially on the suitable choice of the regularization parameter. One extensively studied criterion is the discrepancy principle, see e.g. Morozov [45]. Although this criterion is mathematically rigorous, it requires a reliable estimation of the amount of noise added into the data which may not be available in practical problems. Heuristic approaches are preferable in the case when no *a priori* information about the noise is available. For the Tikhonov zeroth-order regularization method, several heuristic approaches have been proposed, including the L-curve criterion, see Hansen [18], and the generalized cross-validation (GCV), see Wahba [50]. In this paper, we employ the GCV criterion to determine the optimal regularization parameter, λ_{opt}, for the Tikhonov zeroth-order regularization method, namely

$$\lambda_{\text{opt}} : \quad \mathcal{G}(\lambda_{\text{opt}}) = \min_{\lambda > 0} \mathcal{G}(\lambda). \tag{27}$$

Here

$$\mathcal{G}(\cdot) : (0,\infty) \longrightarrow [0,\infty), \quad \mathcal{G}(\lambda) = \frac{\|\mathbf{Ac}_\lambda - \mathbf{b}^\varepsilon\|^2}{[\text{trace}(\mathbf{I}_N - \mathbf{AA}^\dagger)]^2}, \tag{28}$$

where $\mathbf{c}_\lambda$ is given by Eq. (26) with $\mathbf{b} = \mathbf{b}^\varepsilon$.

6. Numerical results and discussion

In this section, we present the performance of the proposed numerical method, namely the alternating iterative MFS described in Sections 3 and 4. To do so, we solve numerically the Cauchy problem given by Eqs. (4a)–(4c) for the two-dimensional Laplace equation in the geometries described below, see also Figs. 1(a)–(d).

6.1. Examples

Example 1 (*Simply connected convex domain with a smooth boundary, see Fig. 1(a)*). We consider the following analytical solution for the temperature

$$u^{(\text{an})}(\mathbf{x}) = x_1^2 - x_2^2, \quad \mathbf{x} = (x_1,x_2) \in \overline{\Omega}, \tag{29a}$$

and the corresponding analytical normal heat flux

$$q^{(\text{an})}(\mathbf{x}) = 2[x_1 n_1(\mathbf{x}) - x_2 n_2(\mathbf{x})], \quad \mathbf{x} = (x_1,x_2) \in \partial\Omega, \tag{29b}$$

in the unit disc $\Omega = \{\mathbf{x} = (x_1,x_2) | \rho(\mathbf{x}) < r\}$, where $\rho(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$ is the radial polar coordinate of $\mathbf{x}$ and $r = 1.0$. Here $\Gamma_1 = \{\mathbf{x} \in \partial\Omega | 0 \leq \theta(\mathbf{x}) \leq 3\pi/2\}$ and $\Gamma_2 = \{\mathbf{x} \in \partial\Omega | 3\pi/2 < \theta(\mathbf{x}) < 2\pi\}$, where $\theta(\mathbf{x})$ is the angular polar coordinate of $\mathbf{x}$.

Example 2 (*Simply connected convex domain with a piecewise smooth boundary, see Fig. 1(b)*). We consider the following analytical solutions for the temperature and the normal heat flux

$$u^{(\text{an})}(\mathbf{x}) = \cos(x_1)\cosh(x_2) + \sin(x_1)\sinh(x_2), \quad \mathbf{x} = (x_1,x_2) \in \overline{\Omega}, \tag{30a}$$

and

$$\begin{aligned} q^{(\text{an})}(\mathbf{x}) = {} & [-\sin(x_1)\cosh(x_2) + \cos(x_1)\sinh(x_2)]n_1(\mathbf{x}) \\ & + [\cos(x_1)\sinh(x_2) + \sin(x_1)\cosh(x_2)]n_2(\mathbf{x}), \quad \mathbf{x} = (x_1,x_2) \in \partial\Omega, \end{aligned} \tag{30b}$$

respectively, in the rectangle $\Omega = (-r,r) \times (-r/2,r/2)$, where $r = 1.0$. Here $\Gamma_1 = \{r\} \times (-r/2,r/2) \cup [-r,r] \times \{\pm r/2\}$ and $\Gamma_2 = \{-r\} \times (-r/2,r/2)$.

Example 3 (*Simply connected concave domain with a smooth boundary, see Fig. 1(c)*). We consider the following analytical solutions for the temperature and the normal heat flux

$$u^{(\text{an})}(\mathbf{x}) = x_1 x_2, \quad \mathbf{x} = (x_1,x_2) \in \overline{\Omega}, \tag{31a}$$

and

$$q^{(\text{an})}(\mathbf{x}) = x_2 n_1(\mathbf{x}) + x_1 n_2(\mathbf{x}), \quad \mathbf{x} = (x_1,x_2) \in \partial\Omega, \tag{31b}$$

respectively, in the epitrochoid $\Omega = \{\mathbf{x} = (x_1,x_2) | \rho(\mathbf{x}) < r(\theta), \theta \in [0,2\pi)\}$.

Here $r(\theta) = \sqrt{(a+b)^2 - 2h(a+b)\cos(a\theta/b) + h^2}$, where $a = 1.0$, $b = 0.25$ and $h = 0.125$, while $\Gamma_1 = \{\mathbf{x} \in \partial\Omega | \rho(\mathbf{x}) = r(\theta), \theta \in [0,3\pi/2]\}$ and $\Gamma_2 = \{\mathbf{x} \in \partial\Omega | \rho(\mathbf{x}) = r(\theta), \theta \in (3\pi/2,2\pi)\}$.

Example 4 (*Doubly connected concave domain with a smooth boundary, see Fig. 1(d)*). We consider the same analytical solutions for the temperature and the normal heat flux as those corresponding to Example 3 in the annular domain $\Omega = \{\mathbf{x} = (x_1,x_2) | r_{\text{int}} < \rho(\mathbf{x}) < r_{\text{out}}\}$, where $r_{\text{int}} = 2.0$ and $r_{\text{out}} = 3.0$. Here $\Gamma_1 = \{\mathbf{x} \in \partial\Omega | \rho(\mathbf{x}) = r_{\text{out}}\}$ and $\Gamma_2 = \{\mathbf{x} \in \partial\Omega | \rho(\mathbf{x}) = r_{\text{int}}\}$.

The inverse problems investigated in this paper have been solved using the uniform distribution of both the MFS boundary collocation points $\mathbf{x}^{(i)}$, $i = 1, \ldots, N$, and the singularities $\boldsymbol{\xi}^{(j)}$, $j = 1, \ldots, M$. Furthermore, the numbers of boundary collocation points N_1 and N_2 corresponding to the over- and under-specified boundaries Γ_1 and Γ_2, respectively, as well as the distance d_{S} between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_{\text{S}}$ on which the singularities are situated, were set to:

(i) $N_1 = 60$, $N_2 = 20$ and $d_{\text{S}} = 3.0$ for Example 1;
(ii) $N_1 = 97$, $N_2 = 19$ and $d_{\text{S}} = 2.0$ in the case of Example 2;
(iii) $N_1 = 60$, $N_2 = 20$ and $d_{\text{S}} = 4.0$ in the case of Example 3; and

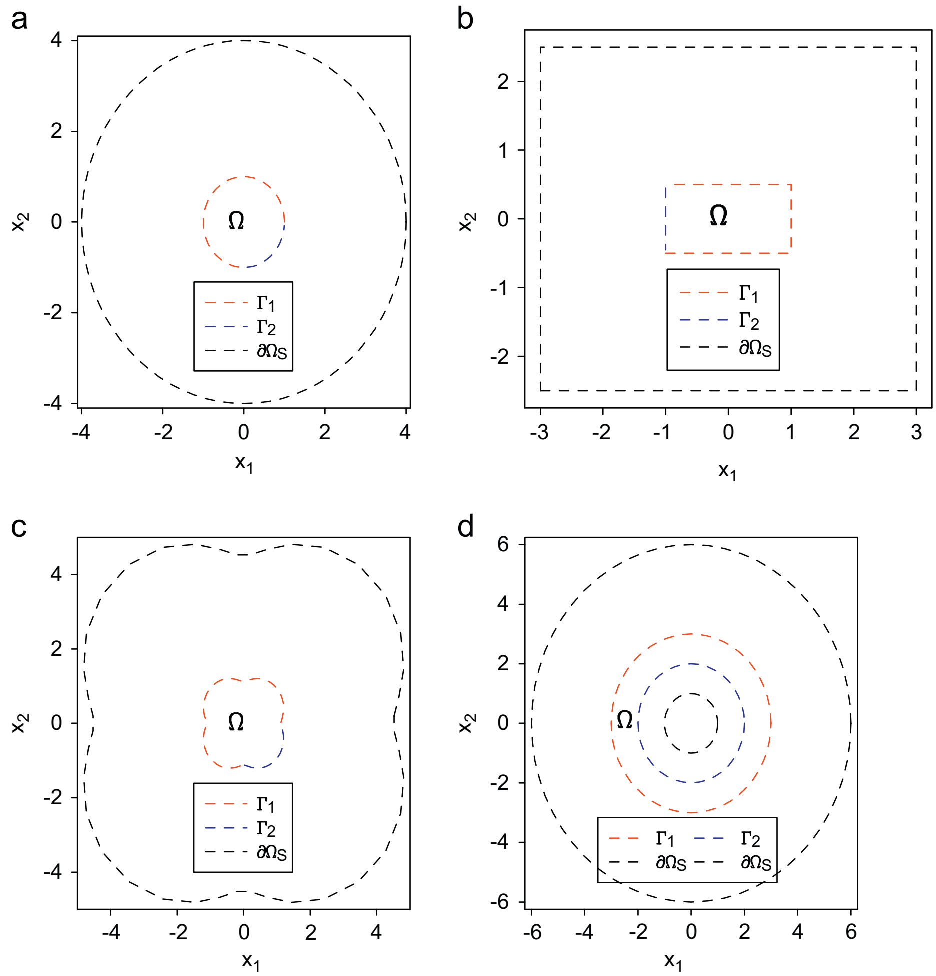

Fig. 1. Schematic diagram of the domain, Ω, over-determined boundary, Γ_1 (–, –, –, in red), under-determined boundary, $\boldsymbol{\Gamma}_2$ (–, –, –, in blue), and pseudo-boundary, $\partial\Omega_S$ (–, –, –), for the inverse problems investigated, namely (a) Example 1 (disc), (b) Example 2 (rectangle), (c) Example 3 (epitrochoid), and (d) Example 4 (annulus), respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(iv) $N_1=60$ and $N_2=40$, while $d_S=1.0$ and $d_S=3.0$ for the inner and outer boundaries, respectively, for Example 4.

For all examples analysed herein the number of singularities was taken to be equal to that of the MFS boundary collocation points, i.e. $M=N=N_1+N_2$. Although not presented herein, it is reported that the numerical results obtained for the unknown temperature and normal heat flux on the boundary Γ_2 are convergent with respect to increasing the distance d_S between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_S$. However, it should be noted that the value $d_S = 1.0$ was found to be sufficiently large such that any further increase of the distance between the singularities and the boundary of the solution domain did not significantly improve the accuracy of the numerical solutions for the examples tested in this paper.

It is also important to mention that for the inverse problems investigated in this paper, as well as the alternating iterative algorithms I and II, the initial guesses $q^{(1)}$ and $u^{(1)}$ for the normal heat flux and temperature, respectively, were taken to be

$$q^{(1)}(\mathbf{x})=0, \quad \mathbf{x}\in\Gamma_2, \tag{32a}$$

and

$$u^{(1)}(\mathbf{x})=0, \quad \mathbf{x}\in\Gamma_2, \tag{32b}$$

respectively. Moreover, all numerical computations have been performed in FORTRAN 90 in double precision on a 3.00 GHz Intel Pentium 4 machine.

6.2. *Results obtained with exact data: convergence of the algorithms*

If N_i MFS collocation points, $\{\mathbf{x}^{(\ell)}\}_{\ell=1}^{N_i}$, are considered on the boundary $\Gamma_i\subset\partial\Omega$ then the *root mean square error* (RMS error) associated with the real valued function $f(\cdot):\Gamma_i\longrightarrow\mathbb{R}$ on Γ_i is defined by

$$\mathrm{RMS}_{\Gamma_i}(f)=\sqrt{\frac{1}{N_i}\sum_{\ell=1}^{N_i} f(\mathbf{x}^{(\ell)})^2}. \tag{33}$$

In order to investigate the convergence of the algorithm, at each iteration, $k\geq 1$, we evaluate the following accuracy errors corresponding to the temperature and normal heat flux on the under-specified boundary, Γ_2, which are defined as *relative RMS errors*, i.e.

$$e_{\mathrm{u}}(k)=\begin{cases}\dfrac{\mathrm{RMS}_{\Gamma_2}(u^{(2k-1)}-u^{(\mathrm{an})})}{\mathrm{RMS}_{\Gamma_2}(u^{(\mathrm{an})})} & \text{for the alternating iterative algorithm with relaxation I,}\\[2ex] \dfrac{\mathrm{RMS}_{\Gamma_2}(u^{(2k)}-u^{(\mathrm{an})})}{\mathrm{RMS}_{\Gamma_2}(u^{(\mathrm{an})})} & \text{for the alternating iterative algorithm with relaxation II,}\end{cases} \tag{34a}$$

and

$$e_q(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_2}(q^{(2k)} - q^{(\mathrm{an})})}{\mathrm{RMS}_{\Gamma_2}(q^{(\mathrm{an})})} & \text{for the alternating iterative algorithm with relaxation I,} \\ \dfrac{\mathrm{RMS}_{\Gamma_2}(q^{(2k-1)} - q^{(\mathrm{an})})}{\mathrm{RMS}_{\Gamma_2}(q^{(\mathrm{an})})} & \text{for the alternating iterative algorithm with relaxation II.} \end{cases} \tag{34b}$$

Here $u^{(2k-1)}$ ($u^{(2k)}$) and $q^{(2k)}$ ($q^{(2k-1)}$) are the temperature and normal heat flux on the boundary Γ_2 retrieved after k iterations using the alternating iterative algorithm with relaxation I (II), respectively, with the mention that each iteration consists of solving two direct well-posed mixed boundary value problems, namely Eqs. (5a)–(5c) and (7a)–(7c) for the alternating iterative algorithm with relaxation I (Eqs. (8a)–(8c) and (10a)–(10c) for the alternating iterative algorithm with relaxation II).

Figs. 2(a) and (b) show, on a logarithmic scale, the accuracy errors e_u and e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, exact Cauchy data and various values of the relaxation parameter ω, in the case of Example 1. It can be seen from these figures that, for all values of the relaxation parameter used in this paper, both errors e_u and e_q keep decreasing until a specific number of iterations, after which the convergence rate of the aforementioned accuracy errors becomes very slow so that they reach a plateau. As expected, for each value of the relaxation parameter employed, $e_u(k) < e_q(k)$ for all $k \geq 1$, i.e. temperatures are more accurate than normal heat fluxes; also, the larger the parameter ω, the lower the number of iterations and, consequently, computational time are required for obtaining accurate numerical results for both the temperature and the normal heat flux on Γ_2. Therefore, choosing $\omega \in (1,2)$ in the alternating iterative algorithms I and II results in a significant reduction of the number of iterations as compared with the corresponding original alternating iterative algorithms proposed by Kozlov et al. [26], i.e. for $\omega = 1$. Furthermore, it can also be noticed from Figs. 2(a) and (b) that, for exact Cauchy data on Γ_1, the errors in the numerical temperature and normal heat flux retrieved on Γ_2 are also decreasing as $\omega \longrightarrow 2$, see e.g. the results obtained for $\omega = 1.90$, while both errors e_u and e_q corresponding to the numerical solutions for the temperature and normal heat flux on Γ_2, obtained using values of the relaxation factor that are not in the vicinity of its maximum admissible value, have almost the same order of magnitude, see e.g. the results obtained for $\omega \in \{0.10, 0.50, 1.00, 1.50\}$.

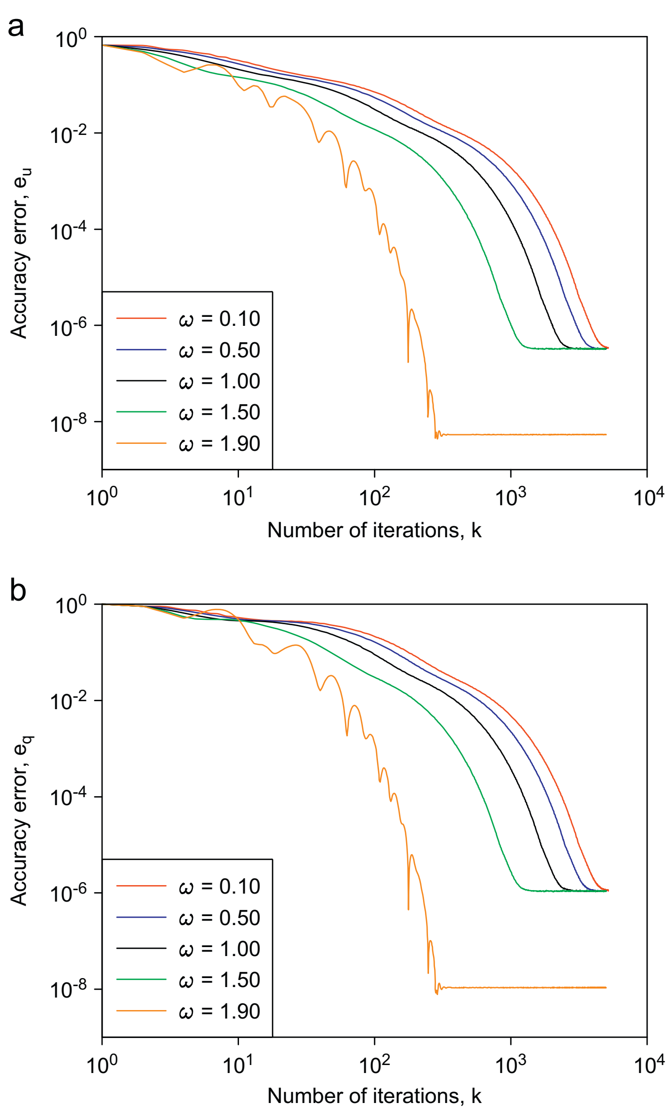

Fig. 2. The accuracy errors (a) e_u, and (b) e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm with relaxation I, exact Cauchy data on Γ_1 and various values of the relaxation parameter, namely $\omega \in \{0.10, 0.50, 1.00, 1.50, 1.90\}$, for Example 1.

The same conclusions can be drawn from Figs. 3(a) and (b), which illustrate the analytical and numerical temperature and normal heat flux, respectively, obtained with $\omega = 1.90$ after $k = 1000$ iterations. From Figs. 2 and 3, it can be concluded that the alternating iterative algorithm with relaxation I described in Section 3 provides excellent approximations for the unknown Dirichlet and Neumann data on Γ_2 and is convergent with respect to increasing the number of iterations, k, if exact Cauchy data are prescribed on the over-specified boundary Γ_1. Although not presented, it is reported that similar results have been obtained for Examples 2–4 and all admissible values of the relaxation parameter, as well as the alternating iterative algorithm with relaxation II applied to all examples investigated in this study.

6.3. *Regularizing stopping criterion*

Once the convergence of the numerical solution to the exact solution, with respect to number of iterations performed, k, has been established, we investigate the stability of the numerical solution for the examples considered. In what follows, the temperature, $u|_{\Gamma_1} = u^{(\mathrm{an})}|_{\Gamma_1}$, and/or the normal heat flux, $q|_{\Gamma_1} = q^{(\mathrm{an})}|_{\Gamma_1}$, on the over-specified boundary have been perturbed as

$$\tilde{u}^{\varepsilon}|_{\Gamma_1} = u|_{\Gamma_1} + \delta u, \quad \delta u = \mathtt{G05DDF}(0, \sigma_u), \quad \sigma_u = \max_{\Gamma_1} |u| \times (p_u/100), \tag{35}$$

and

$$\tilde{q}^{\varepsilon}|_{\Gamma_1} = q|_{\Gamma_1} + \delta q, \quad \delta q = \mathtt{G05DDF}(0, \sigma_q), \quad \sigma_q = \max_{\Gamma_1} |q| \times (p_q/100), \tag{36}$$

respectively. Here δu and δq are Gaussian random variables with mean zero and standard deviations σ_u and σ_q, respectively, generated by the NAG subroutine G05DDF [46], while $p_u\%$ and $p_q\%$ are the percentages of additive noise included into the input boundary temperature, $u|_{\Gamma_1}$, and normal heat flux, $q|_{\Gamma_1}$, respectively, in order to simulate the inherent measurement errors.

The evolution of the accuracy errors, e_u and e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $p_q = 5\%$ noise added into the Neumann data

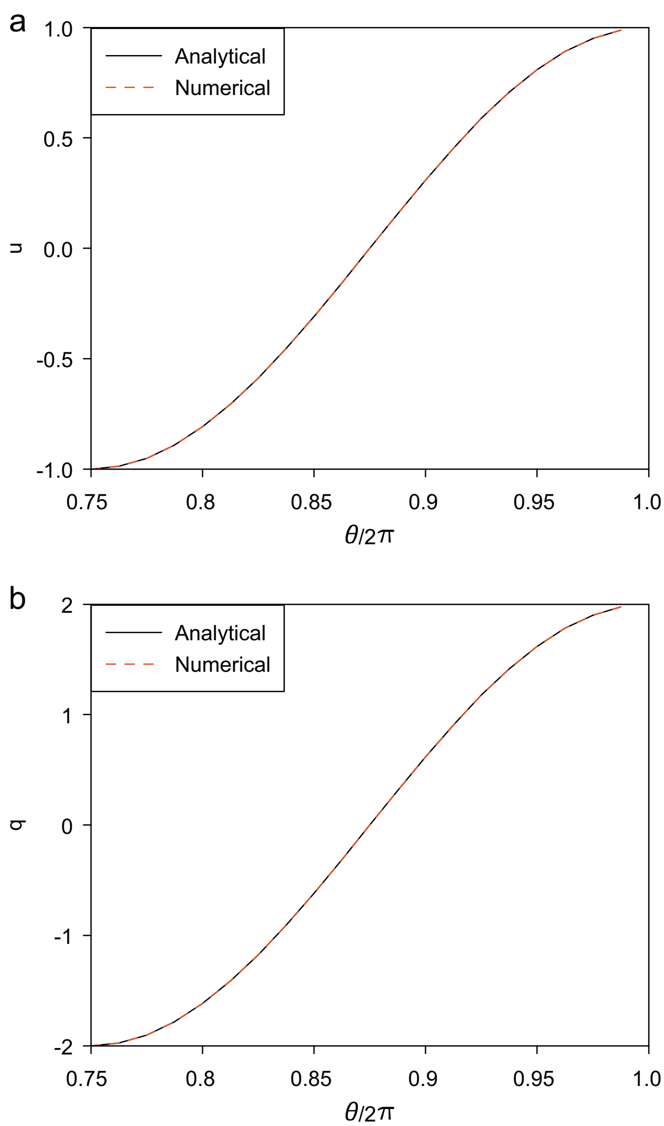

Fig. 3. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, exact Cauchy data on Γ_1, $\omega = 1.90$ and $k = 1000$ iterations, for Example 1.

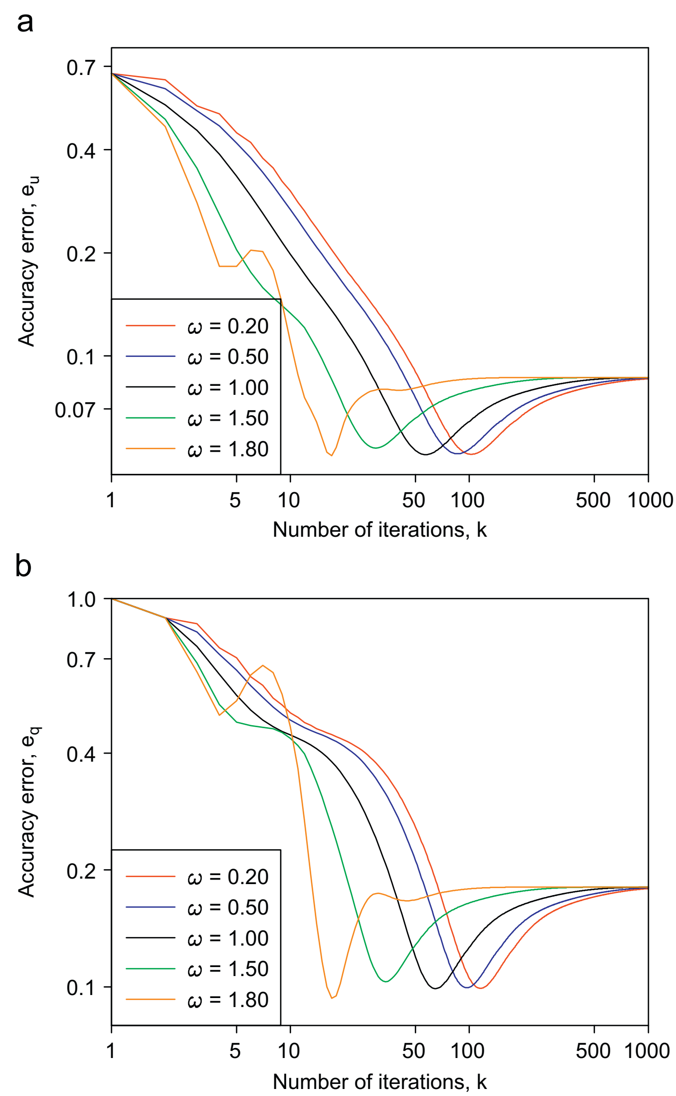

Fig. 4. The accuracy errors (a) e_u, and (b) e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $p_q = 5\%$ noise added into the Neumann data on Γ_1 and various values of the relaxation parameter, ω, namely $\omega \in \{0.20, 0.50, 1.00, 1.50, 1.80\}$, for Example 1.

on Γ_1 and various values of the relaxation parameter, ω, for Example 1, are displayed, on a logarithmic scale, in Figs. 4(a) and (b), respectively. From these figures it can be noted that the number of iterations required for both errors e_u and e_q to attain their corresponding minimum values (i.e. to obtain the optimal numerical solution to the Cauchy problem) decreases with respect to increasing the value of the relaxation parameter, ω. Similar to the case of exact Cauchy data on Γ_1, both e_u and e_q are slightly decreasing as $\omega \longrightarrow 2$, see e.g. the results obtained for $\omega = 1.80$, while the inaccuracies in the numerical solutions for both the temperature and normal heat flux on the boundary Γ_1, obtained using values of the relaxation parameters that are not close to its maximum admissible value, have almost the same order of magnitude.

Figs. 5(a) and (b) present, on a logarithmic scale, the accuracy errors e_u and e_q, respectively, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of Gaussian random noise $p_u \in \{1\%, 5\%, 10\%\}$ added into the temperature data $u|_{\Gamma_1}$, for the Cauchy problem given by Example 4. From these figures it can be seen that, for each fixed value of p_u, the errors in predicting the temperature and normal heat flux on the under-specified boundary Γ_2 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion must be used in order to cease the iterative process at the point where the errors in the numerical solutions start increasing.

To define the stopping criterion required for regularizing/stabilizing the iterative methods analysed in this paper, after each iteration, k, we evaluate the following convergence error which is associated with the temperature on the over-specified boundary, Γ_1, namely

$$E(k) = \begin{cases} \mathrm{RMS}_{\Gamma_1}(u^{(2k-1)} - \tilde{u}^{\varepsilon}) / \mathrm{RMS}_{\Gamma_1}(\tilde{u}^{\varepsilon}) & \text{for the alternating iterative algorithm with relaxation I,} \\ \mathrm{RMS}_{\Gamma_1}(u^{(2k)} - \tilde{u}^{\varepsilon}) / \mathrm{RMS}_{\Gamma_1}(\tilde{u}^{\varepsilon}) & \text{for the alternating iterative algorithm with relaxation II.} \end{cases} \qquad (37)$$

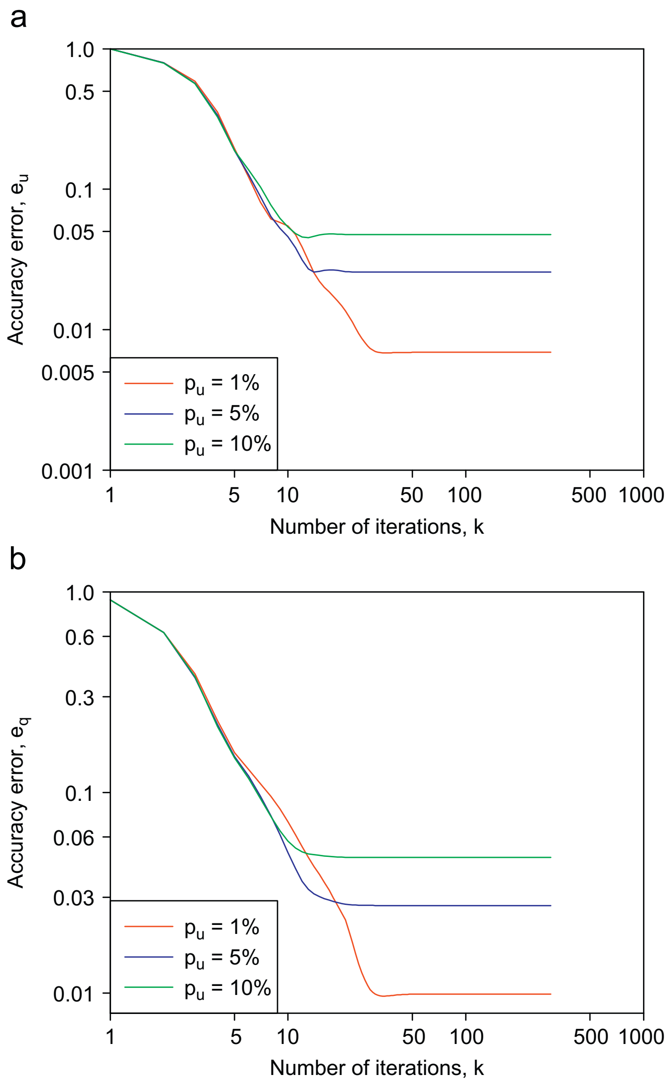

Fig. 5. The accuracy errors (a) e_u, and (b) e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 5\%, 10\%\}$, for Example 4.

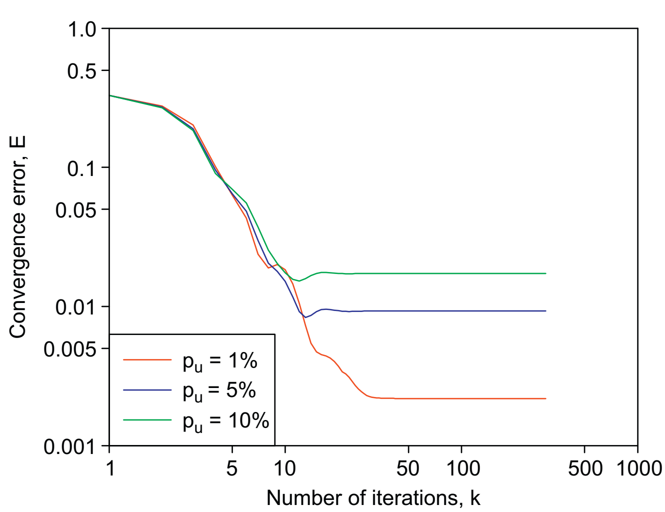

Fig. 6. The convergence error, E, as a function of the number of iterations, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 5\%, 10\%\}$, for Example 4.

Here $u^{(2k-1)}$ ($u^{(2k)}$) is the temperature on the boundary Γ_1, retrieved numerically after k iterations by solving the well-posed mixed direct boundary value problem (5a)–(5c) [(10a)–(10c)], in the case of the alternating iterative algorithm I (II), while $\tilde{u}^{\varepsilon}$ is the perturbed Dirichlet data (boundary temperature) on the over-specified boundary Γ_1, as given by Eq. (35). This error E should tend to zero as the sequences $\{u^{(2k-1)}\}_{k \geq 1}$ and $\{u^{(2k)}\}_{k \geq 1}$ tend to the analytical solution, $u^{(\mathrm{an})}$, in the space $H^1(\Omega)$ and hence it is expected to provide an appropriate stopping criterion. Indeed, if we investigate the error E obtained at each iteration for Example 4, using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of Gaussian random noise $p_u \in \{1\%, 5\%, 10\%\}$ added into the temperature data $u|_{\Gamma_1}$, we obtain the curves graphically represented in Fig. 6. By comparing Figs. 5 and 6, it can be noticed that the convergence error E, as well as the accuracy errors e_u and e_q, attain their corresponding minimum at around the same number of iterations. Therefore, for noisy Cauchy data a natural stopping criterion ceases the MFS alternating iterative algorithms with relaxation I and II at the optimal number of iterations, k_{opt}, given by

$$k_{\mathrm{opt}}: \quad E(k_{\mathrm{opt}}) = \min_{k \geq 1} E(k). \tag{38}$$

Although not illustrated, it is important to mention that similar results and conclusions have been obtained for the other examples considered and $\omega \in (0,2)$.

As mentioned in the previous section, for exact data the iterative process is convergent with respect to increasing the number of iterations, k, since the accuracy errors e_u and e_q keep decreasing even after a large number of iterations, see e.g. Fig. 2 corresponding to Example 1. It should be noted in this case that a stopping criterion is not necessary since the numerical solution is convergent with respect to increasing the number of iterations. However, even in this case the errors E, e_u and e_q have a similar behaviour and the error E may be used to stop the iterative process at the point where the rate of convergence is very small and no substantial improvement in the numerical solution is obtained even if the iterative process is continued. Therefore, it can be concluded that the regularizing stopping criterion (38) proposed for the alternating iterative algorithms with relaxation I and II is very efficient in locating the point where the errors start increasing and the iterative process should be ceased.

6.4. *Results obtained with noisy data: stability of the algorithms*

Based on the stopping criterion (38) described in Section 6.3, the analytical and numerical values for the temperature, u, and normal heat flux, q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the temperature data on the over-specified boundary Γ_1, for Example 1, are illustrated in Figs. 7(a) and (b), respectively. From these figures it can be seen that the numerical solution is a stable approximation for the exact solution, free of unbounded and rapid oscillations. It should also be noted from Figs. 7(a) and (b) that the numerical solution converges to the exact solution as the level of noise, p_u, added into the input Dirichlet data decreases.

The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{\mathrm{opt}})$ and $e_q(k_{\mathrm{opt}})$, and the CPU time, obtained using the alternating iterative algorithm I, the stopping criterion (38), various levels of noise added into the Dirichlet data on Γ_1 and various values of the relaxation parameter, $\omega \in (0,2)$, for the Cauchy problem given by Example 1, are presented in Table 1. The following major conclusions can be drawn from this table:

(i) For all fixed values of the relaxation parameter $\omega \in (0,2)$, both accuracy errors $e_u(k_{\mathrm{opt}})$ and $e_q(k_{\mathrm{opt}})$ decrease as p_u decreases

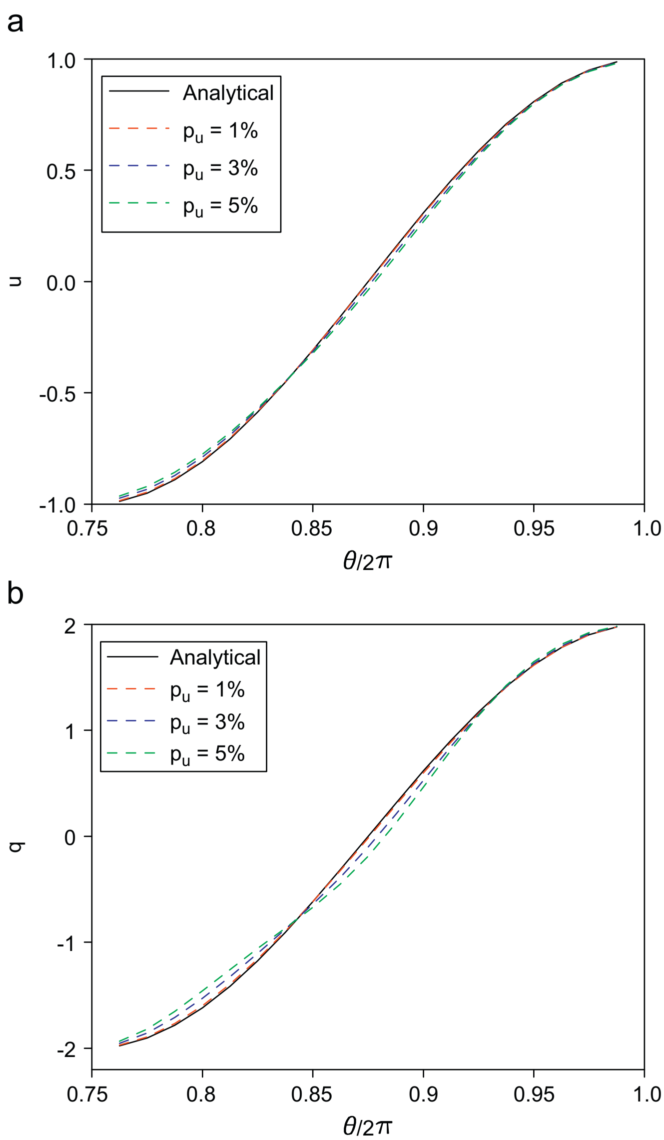

Fig. 7. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 1.

(i.e. the algorithm I is stable with respect to decreasing the level of noise added into the Dirichlet data on Γ_1), while the optimal number of iterations k_{opt} and, consequently, the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 increase as p_u decreases.

(ii) For all fixed amounts of noise added into the temperature on the over-specified boundary Γ_1, $p_u \in \{1\%, 3\%, 5\%\}$, the accuracy errors $e_u(k_{opt})$ and $e_q(k_{opt})$, the optimal number of iterations, k_{opt} and the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 decrease as $\omega \longrightarrow 2$, i.e. as more over-relaxation is introduced in the algorithm I. However, it should be stressed out that the differences, in terms of accuracy, between the numerical results for both $u|_{\Gamma_2}$ and $q|_{\Gamma_2}$, obtained for various values of the relaxation parameter, ω, are not very significant.

Table 1
The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the computational time, obtained using the alternating iterative algorithm I, the regularizing stopping criterion (38), various amounts of noise added into $u|_{\Gamma_1}$, i.e. $p_u \in \{1\%, 3\%, 5\%\}$ and $p_q = 0\%$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	p_u (%)	p_q (%)	k_{opt}	$e_u(k_{opt})$	$e_q(k_{opt})$	CPU time (s)
0.10	1	0	619	0.54281×10^{-2}	0.92962×10^{-2}	3358.67
	3	0	401	0.21528×10^{-1}	0.42797×10^{-1}	2161.21
	5	0	288	0.37941×10^{-1}	0.77268×10^{-1}	1521.75
0.50	1	0	486	0.54322×10^{-2}	0.93019×10^{-2}	2638.01
	3	0	318	0.21565×10^{-1}	0.42946×10^{-1}	1697.76
	5	0	228	0.37961×10^{-1}	0.77372×10^{-1}	1201.70
1.00	1	0	327	0.54264×10^{-2}	0.92927×10^{-2}	1726.57
	3	0	212	0.21537×10^{-1}	0.42834×10^{-1}	1122.70
	5	0	151	0.37933×10^{-1}	0.77146×10^{-1}	798.75
1.50	1	0	156	0.53938×10^{-2}	0.91168×10^{-2}	826.04
	3	0	104	0.21229×10^{-1}	0.41895×10^{-1}	550.37
	5	0	94	0.36700×10^{-1}	0.74137×10^{-1}	485.59
1.80	1	0	73	0.36273×10^{-2}	0.53332×10^{-2}	385.53
	3	0	33	0.99652×10^{-2}	0.15014×10^{-1}	172.26
	5	0	32	0.26705×10^{-1}	0.49883×10^{-1}	166.85

In order to assess the performance of the alternating iterative algorithm I with under-, no and over-relaxation, we exemplify by considering Example 1 with $p_u = 1\%$: In this case, the CPU times needed for the alternating iterative algorithm I with $\omega = 0.50$ (under-relaxation), $\omega = 1.00$ (no relaxation) and $\omega = 1.50$ (over-relaxation) to reach the numerical solutions for the temperature and normal heat flux on Γ_2 were found to be 2638.01, 1726.57 and 826.04 s, respectively, while the corresponding values for the optimal iteration number required, k_{opt}, were found to be 486, 327 and 156, respectively. This means that, to attain the numerical solutions for the unknown Dirichlet and Neumann data on Γ_2, the alternating iterative algorithm I with over-relaxation ($\omega = 1.50$) requires a reduction in the number of iterations performed and CPU time by approximately 52% and 68% with respect to those corresponding to the standard iterative algorithm I as proposed by Kozlov et al. [26], i.e. without relaxation ($\omega = 1.00$), and the alternating iterative algorithm I with under-relaxation ($\omega = 0.50$), respectively.

Similar conclusions to those obtained from Figs. 7(a) and (b) can be drawn from Figs. 8(a) and (b), which present the numerical values for the temperature and normal heat flux obtained on the under-specified boundary Γ_2, in comparison with their analytical counterparts, using the alternating iterative algorithm I, the regularizing stopping criterion (38), $\omega = 1.50$ and various amounts of noise added into the normal heat flux $q|_{\Gamma_1}$, i.e. $p_q \in \{1\%, 3\%, 5\%\}$, for Example 1. By comparing Figs. 7 and 8, it can be observed that, as expected, the alternating iterative algorithm I applied to Example 1 is more sensitive to noise added into the normal heat flux $q|_{\Gamma_1}$ than to perturbations of the temperature $u|_{\Gamma_1}$ since the former contains first-order derivatives of the latter.

Table 2 tabulates the values of the optimal iteration number, k_{opt}, according to the stopping criterion (38), the corresponding accuracy errors given by Eqs. (34a)–(34b), and the CPU time, obtained using the alternating iterative algorithm I, various levels of noise added into the Neumann data on Γ_1 and various values of the relaxation parameter, $\omega \in (0,2)$, for the Cauchy problem given by Example 1. From Tables 1 and 2 it can be noticed that the sensitivity of the alternating iterative algorithm I with respect to

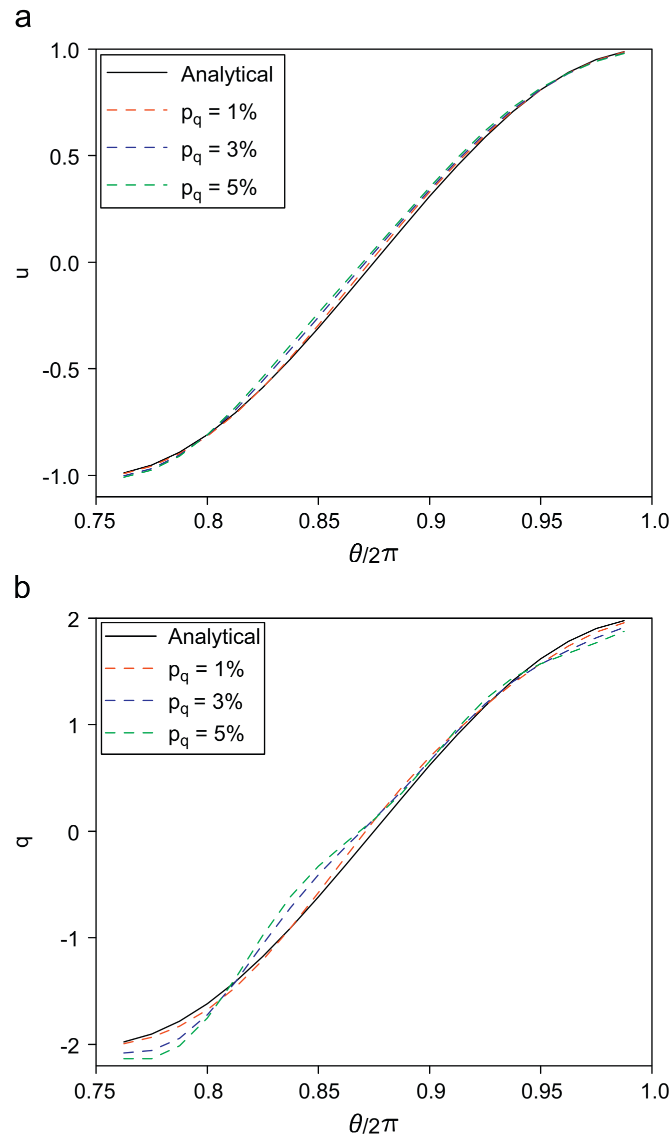

Fig. 8. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%, 3\%, 5\%\}$, for Example 1.

noisy Dirichlet and Neumann data on Γ_1, for Example 1, results in the following:

(i) More inaccurate numerical results for both $u|_{\Gamma_2}$ and $q|_{\Gamma_2}$ are obtained for perturbed normal heat flux on Γ_1 than for noisy temperature on Γ_1.
(ii) The optimal number of iterations k_{opt} and hence the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_2 for perturbed temperature on Γ_1 are, in general, larger that those corresponding to noisy normal heat flux on Γ_1.

The same conclusions, as those drawn from Tables 1 and 2, regarding the stability of the numerical results obtained using the alternating iterative algorithm I with relaxation with respect to the level of noise added into the Cauchy data, and the sensitivity of the optimal number of iterations performed and, consequently, the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_2, remain valid also if both the Dirichlet and Neumann data on Γ_1 are perturbed by noise and these are presented in Table 3. The analytical and numerical values for the temperature, $u|_{\Gamma_2}$, and normal heat flux, $q|_{\Gamma_2}$, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and $p_u = p_q \in \{1\%, 3\%, 5\%\}$, for Example 1, are shown in Figs. 9(a) and (b), respectively. We can conclude from these figures that stable

Table 2
The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the computational time, obtained using the alternating iterative algorithm I, the regularizing stopping criterion (38), various amounts of noise added into $q|_{\Gamma_1}$, i.e. $p_u = 0\%$ and $p_q \in \{1\%, 3\%, 5\%\}$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	p_u (%)	p_q (%)	k_{opt}	$e_u(k_{opt})$	$e_q(k_{opt})$	CPU time (s)
0.20	0	1	2712	0.17303×10^{-1}	0.36123×10^{-1}	14334.21
	0	3	147	0.37652×10^{-1}	0.74374×10^{-1}	778.06
	0	5	102	0.51636×10^{-1}	0.10358×10^{0}	539.12
0.50	0	1	2256	0.17303×10^{-1}	0.36123×10^{-1}	11969.48
	0	3	123	0.37281×10^{-1}	0.73587×10^{-1}	649.92
	0	5	86	0.51805×10^{-1}	0.10351×10^{0}	453.42
1.00	0	1	1505	0.17303×10^{-1}	0.36123×10^{-1}	7958.92
	0	3	82	0.37811×10^{-1}	0.74868×10^{-1}	443.25
	0	5	57	0.51531×10^{-1}	0.10367×10^{0}	298.96
1.50	0	1	214	0.16612×10^{-1}	0.36080×10^{-1}	1159.87
	0	3	42	0.38289×10^{-1}	0.76983×10^{-1}	225.34
	0	5	30	0.53806×10^{-1}	0.10899×10^{0}	162.28
1.80	0	1	84	0.16609×10^{-1}	0.36073×10^{-1}	450.62
	0	3	18	0.39436×10^{-1}	0.82152×10^{-1}	92.65
	0	5	17	0.51142×10^{-1}	0.93332×10^{-1}	90.18

Table 3
The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the computational time, obtained using the alternating iterative algorithm I, the regularizing stopping criterion (38), various amounts of noise added into the Cauchy data $u|_{\Gamma_1}$ and $q|_{\Gamma_1}$, i.e. $p_u, p_q \in \{1\%, 3\%, 5\%\}$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	p_u (%)	p_q (%)	k_{opt}	$e_u(k_{opt})$	$e_q(k_{opt})$	CPU time (s)
0.50	1	1	1027	0.71000×10^{-2}	0.21665×10^{-1}	5472.79
	1	3	185	0.18098×10^{-1}	0.41589×10^{-1}	971.34
	1	5	126	0.32464×10^{-1}	0.67204×10^{-1}	660.56
	3	1	236	0.13052×10^{-1}	0.30916×10^{-1}	1246.04
	3	3	166	0.20953×10^{-1}	0.49745×10^{-1}	868.96
	3	5	108	0.37518×10^{-1}	0.70924×10^{-1}	583.56
	5	1	208	0.28335×10^{-1}	0.62977×10^{-1}	1091.03
	5	3	140	0.32643×10^{-1}	0.60176×10^{-1}	731.10
	5	5	90	0.49354×10^{-1}	0.96463×10^{-1}	477.17
1.00	1	1	686	0.71000×10^{-2}	0.21664×10^{-1}	3613.28
	1	3	123	0.18121×10^{-1}	0.50124×10^{-1}	652.39
	1	5	84	0.32410×10^{-1}	0.67146×10^{-1}	440.59
	3	1	157	0.12907×10^{-1}	0.30292×10^{-1}	830.95
	3	3	112	0.21329×10^{-1}	0.42030×10^{-1}	585.70
	3	5	72	0.37384×10^{-1}	0.70789×10^{-1}	380.37
	5	1	139	0.28236×10^{-1}	0.62662×10^{-1}	733.12
	5	3	94	0.33087×10^{-1}	0.61075×10^{-1}	491.09
	5	5	60	0.49101×10^{-1}	0.96181×10^{-1}	318.57
1.50	1	1	345	0.71000×10^{-2}	0.21662×10^{-1}	1830.73
	1	3	61	0.18100×10^{-1}	0.49164×10^{-1}	320.32
	1	5	42	0.34582×10^{-1}	0.67966×10^{-1}	230.35
	3	1	79	0.12677×10^{-1}	0.29744×10^{-1}	412.71
	3	3	56	0.21754×10^{-1}	0.42368×10^{-1}	303.82
	3	5	36	0.40714×10^{-1}	0.74853×10^{-1}	187.03
	5	1	70	0.27742×10^{-1}	0.61195×10^{-1}	368.18
	5	3	48	0.33571×10^{-1}	0.62095×10^{-1}	250.32
	5	5	32	0.51137×10^{-1}	0.99515×10^{-1}	165.20

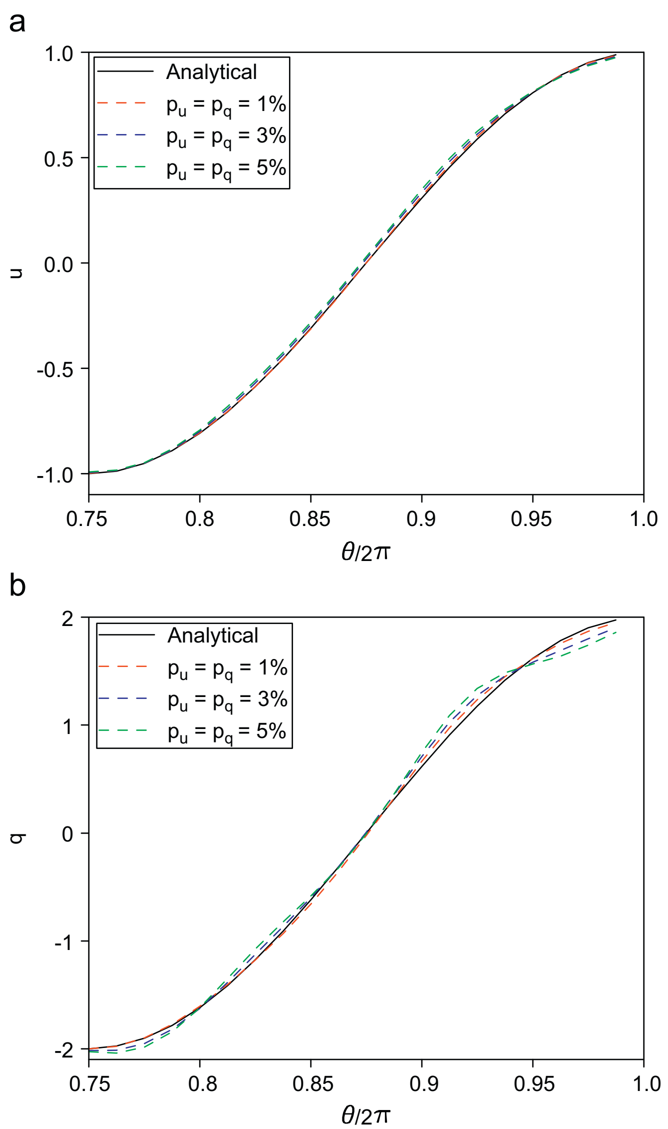

Fig. 9. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega=1.50$ and various levels of noise added into both the Dirichlet and the Neumann data on Γ_1, namely $p_u=p_q\in\{1\%,3\%,5\%\}$, for Example 1.

Table 4

The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the computational time, obtained using the alternating iterative algorithm II, the regularizing stopping criterion (38), various amounts of noise added into $u|_{\Gamma_1}$, i.e. $p_u\in\{1\%,3\%,5\%\}$ and $p_q=0\%$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	p_u (%)	p_q (%)	k_{opt}	$e_u(k_{opt})$	$e_q(k_{opt})$	CPU time (s)
0.10	1	0	623	0.54361×10^{-2}	0.93071×10^{-2}	3362.93
	3	0	385	0.22545×10^{-1}	0.45549×10^{-1}	2079.95
	5	0	285	0.38194×10^{-1}	0.79425×10^{-1}	1546.75
0.50	1	0	489	0.54413×10^{-2}	0.93141×10^{-2}	2634.54
	3	0	303	0.22559×10^{-1}	0.45553×10^{-1}	1639.37
	5	0	266	0.38938×10^{-1}	0.79975×10^{-1}	1444.64
1.00	1	0	329	0.54353×10^{-2}	0.93070×10^{-2}	1764.76
	3	0	203	0.22568×10^{-1}	0.45621×10^{-1}	1097.26
	5	0	151	0.38179×10^{-1}	0.79401×10^{-1}	821.40
1.50	1	0	169	0.54077×10^{-2}	0.92723×10^{-2}	905.82
	3	0	109	0.21857×10^{-1}	0.44030×10^{-1}	605.76
	5	0	79	0.37893×10^{-1}	0.77958×10^{-1}	400.17
1.80	1	0	44	0.31226×10^{-2}	0.45855×10^{-2}	234.50
	3	0	35	0.99887×10^{-2}	0.13793×10^{-1}	186.35
	5	0	34	0.26742×10^{-1}	0.51738×10^{-1}	181.01

Table 5

The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the computational time, obtained using the alternating iterative algorithm II, the regularizing stopping criterion (38), various amounts of noise added into $q|_{\Gamma_1}$, i.e. p_u=0% and $p_q\in\{1\%,3\%,5\%\}$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	p_u (%)	p_q (%)	k_{opt}	$e_u(k_{opt})$	$e_q(k_{opt})$	CPU time (s)
0.20	0	1	2712	0.17303×10^{-1}	0.36123×10^{-1}	14416.84
	0	3	157	0.37734×10^{-1}	0.74557×10^{-1}	858.67
	0	5	109	0.52036×10^{-1}	0.10410×10^{0}	575.31
0.50	0	1	2265	0.17303×10^{-1}	0.36123×10^{-1}	11879.26
	0	3	132	0.37845×10^{-1}	0.74412×10^{-1}	686.39
	0	5	92	0.52116×10^{-1}	0.10353×10^{0}	484.29
1.00	0	1	1511	0.17303×10^{-1}	0.36123×10^{-1}	8038.51
	0	3	88	0.37314×10^{-1}	0.73381×10^{-1}	460.82
	0	5	61	0.51861×10^{-1}	0.10322×10^{0}	322.15
1.50	0	1	217	0.16612×10^{-1}	0.36080×10^{-1}	1147.50
	0	3	45	0.37569×10^{-1}	0.73196×10^{-1}	244.48
	0	5	36	0.51668×10^{-1}	0.10092×10^{0}	183.39
1.80	0	1	113	0.16612×10^{-1}	0.36080×10^{-1}	596.75
	0	3	19	0.41349×10^{-1}	0.82088×10^{-1}	98.28
	0	5	18	0.59205×10^{-1}	0.11916×10^{0}	95.53

numerical solutions for the unknown temperature and normal heat flux on Γ_2, free of unbounded and rapid oscillations, are obtained also when both the Dirichlet and Neumann data on Γ_1 are noisy.

Accurate, convergent and stable numerical results for both the temperature and the normal heat flux on Γ_2 have also been obtained in the case of the Cauchy problem associated with Example 1, when using the alternating iterative algorithm II, various values for the relaxation parameter, $\omega\in(0,2)$, and various amounts of noise added into the Dirichlet or Neumann data on the over-specified boundary Γ_1. The quantitative results, obtained for the alternating iterative algorithm II, various values for the relaxation parameter and various levels of noise added into the boundary temperature and normal heat flux data on Γ_1, are tabulated in Tables 4 and 5, respectively, in terms of the optimal iteration number, k_{opt}, and the accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$. From these tables one can draw similar conclusions regarding the sensitivity of the number of iterations performed and corresponding accuracy errors as functions of the relaxation parameter to those obtained for the alternating iterative algorithm I and displayed in Tables 1 and 2.

The performance of the alternating iterative algorithm II with under-, no and over-relaxation is exemplified by considering Example 1 with p_q=1%: In this case, the CPU times needed for the alternating iterative algorithm II with $\omega=0.50$ (under-relaxation), $\omega=1.00$ (no relaxation) and $\omega=1.50$ (over-relaxation) to reach the numerical solutions for $u|_{\Gamma_2}$ and $q|_{\Gamma_2}$ were found to be 11879.26, 8038.51 and 1147.50 s, respectively, while the corresponding values for the optimal iteration number required, k_{opt}, were found to be 2265, 1511 and 217, respectively. This means that, in order to attain the numerical solutions for the unknown Dirichlet and Neumann data on Γ_2, the alternating iterative algorithm II with over-relaxation ($\omega=1.50$) requires a reduction in the number of iterations performed, as well as CPU time, by approximately 85% and 90% with respect to those corresponding to the standard iterative algorithm II as proposed by Kozlov et al.

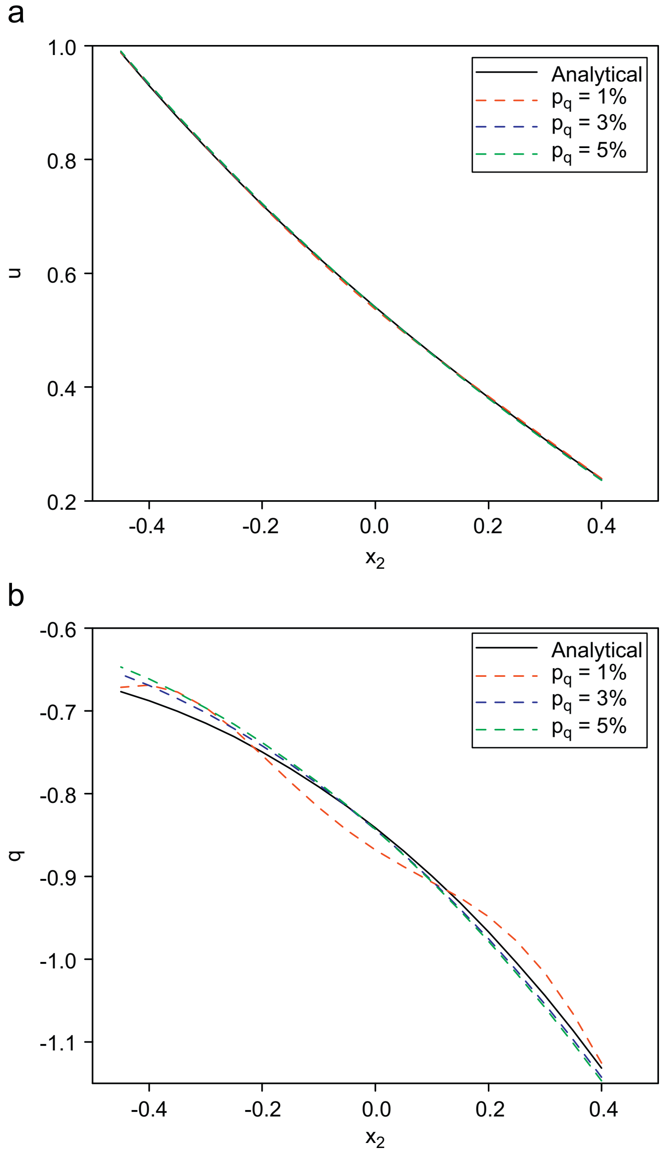

Fig. 10. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega=1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%,3\%,5\%\}$, for Example 2.

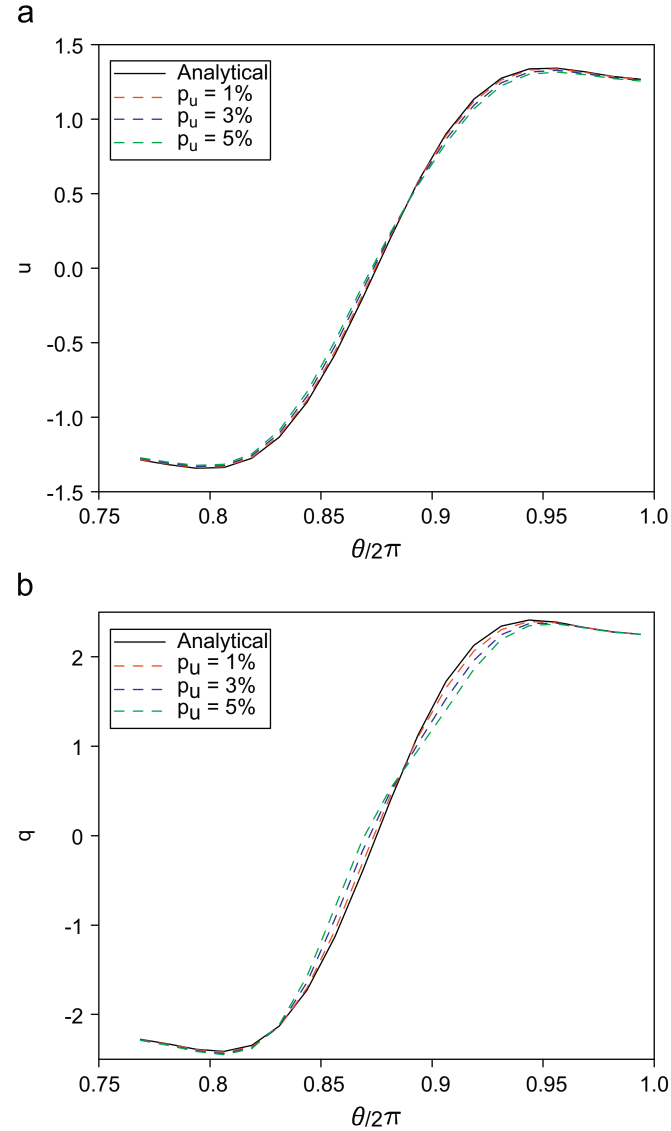

Fig. 11. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm II, $\omega=1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%,3\%,5\%\}$, for Example 3.

[26], i.e. without relaxation ($\omega=1.00$), and the alternating iterative algorithm II with under-relaxation ($\omega=0.50$), respectively.

The proposed MFS-alternating iterative algorithm I, in conjunction with the stopping criterion (38), works equally well also for the Cauchy problem (4a)–(4c) associated with the Laplace equation in a simply connected convex two-dimensional domain with a piecewise smooth boundary, such as the rectangle investigated in Example 2. Figs. 10(a) and (b) show the numerical results for the temperature and normal heat flux on the boundary Γ_2, obtained using the stopping criterion (38), $M=N=116$, $\omega=1.50$ and various amounts of noise added into the Neumann data, namely $p_q \in \{1\%,3\%,5\%\}$, in comparison with their corresponding analytical values, in the case of Example 2.

Similar stable numerical results for both the unknown temperature, $u|_{\Gamma_2}$, and normal heat flux, $q|_{\Gamma_2}$, which are at the same time free of unbounded and rapid oscillations, have been obtained, using the alternating iterative algorithm II, $M=N=80$, $\omega=1.50$ and $p_u \in \{1\%,3\%,5\%\}$, for the two-dimensional steady-state isotropic heat conduction Cauchy problem (4a)–(4c) in a simply connected concave domain with a smooth boundary, such as the epitrochoid considered in Example 3, and these are illustrated in Figs. 11(a) and (b), respectively. The same conclusions have been obtained when solving the Cauchy problem (4a)–(4b) corresponding to the Laplace equation in a doubly connected concave domain with a smooth boundary, namely the annular domain considered in Example 4, by employing the alternating iterative algorithm II, $M=N=100$, $\omega=1.50$ and noisy Neumann data ($p_q \in \{1\%,5\%,10\%\}$), while the analytical and numerical results for the unknown temperature, $u|_{\Gamma_2}$, and normal heat flux, $q|_{\Gamma_2}$, are displayed in Figs. 12(a) and (b), respectively.

From the numerical results presented in this section, it can be concluded that the stopping criterion developed in Section 6.3 has a regularizing effect and the numerical solution obtained by the iterative MFS described in this paper is convergent and stable with respect to increasing the number of MFS boundary collocation points and decreasing the level of noise added into the Cauchy input data, respectively.

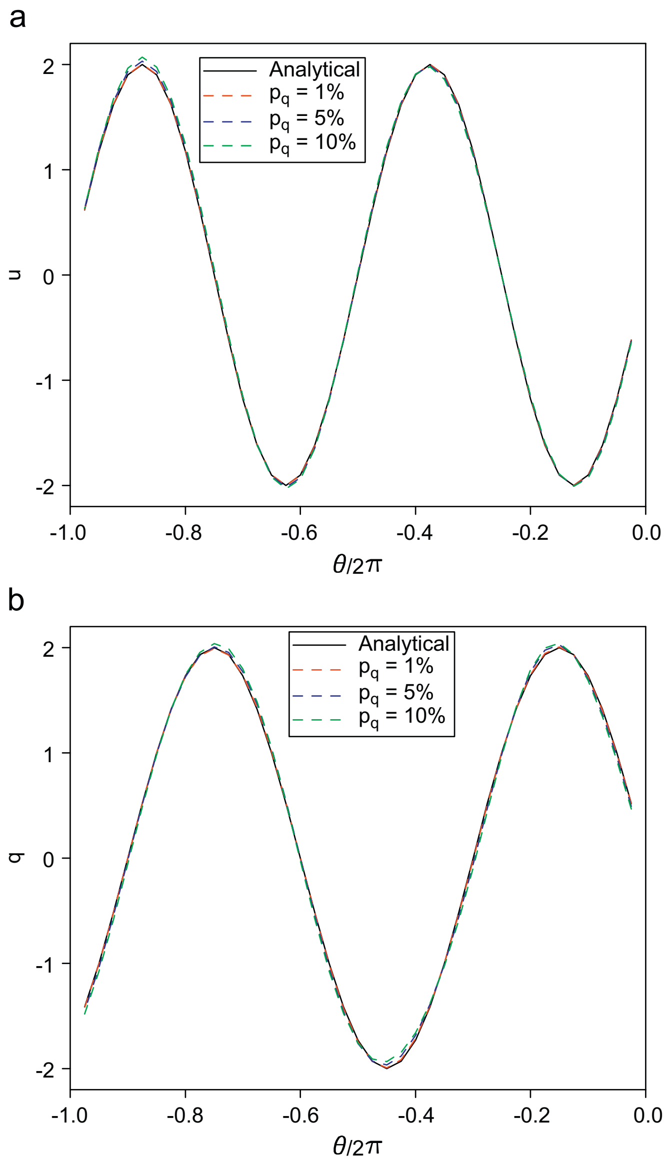

Fig. 12. The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%, 5\%, 10\%\}$, for Example 4.

7. Conclusions

In this paper, we proposed two algorithms involving the relaxation of either the given Dirichlet data (temperature) or the prescribed Neumann data (normal heat flux) on the over-specified boundary in the case of the alternating iterative algorithm of Kozlov et al. [26] applied to two-dimensional steady-state isotropic heat conduction Cauchy problems. The two mixed, well-posed and direct problems corresponding to each iteration of the numerical procedure were solved using a meshless method, namely the MFS, in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter was selected according to the GCV criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, was also presented. The MFS-based iterative algorithms with relaxation were tested for Cauchy problems associated with the Laplace operator in simply and doubly connected, convex and concave domains, with smooth or piecewise smooth boundaries. The numerical results obtained using these procedures that the proposed methods are consistent, accurate, convergent with respect to increasing the number of MFS boundary collocation points and stable with respect to decreasing the amount of noise added into the Cauchy data. One possible disadvantage of the MFS-based iterative algorithms is related to the optimal choice of the regularization parameter associated with the Tikhonov regularization method which requires, at each step of the alternating iterative algorithm of Kozlov et al. [26], additional iterations with respect to the regularization parameter. However, this inconvenience was overcome by employing the relaxation procedures presented in this study, emphasizing at the same time the computational efficiency of the relaxation procedures applied to the alternating iterative algorithm of Kozlov et al. [26].

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