MFS iterative algorithms with relaxation for the stable temperature reconstruction in two-dimensional steady-state isotropic and anisotropic heat conduction problems from incomplete boundary data

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Abstract. We study the stable reconstruction of the boundary and internal temperatures for two-dimensional steady-state (an)isotropic heat conduction problems from prescribed noisy Cauchy data on a part of the boundary. Two MFS-based iterative algorithms involving the relaxation of either the given boundary temperature (Dirichlet data) or the prescribed normal flux (Neumann data) on the over-specified boundary are employed.

Mathematical Formulation of the Problem

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ occupied by a solid characterised by the homogeneous, symmetric and positive-definite thermal conductivity tensor $\mathbb{K} = \left[\mathrm{K}_{ij}\right]_{1\le i,j\le 2}$. We also assume that Ω is bounded by a (piecewise) smooth curve $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \varnothing$, $\Gamma_2 \neq \varnothing$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$.

Let $\mathrm{H}^1(\Omega)$ be the Sobolev space of real-valued functions in Ω endowed with the standard norm. We denote by $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}^1_{\Gamma_i}(\Omega)$, $i = 1, 2$, the subspaces of functions from $\mathrm{H}^1(\Omega)$ that vanish on $\partial\Omega$ and Γ_i, $i = 1, 2$, respectively. Herein, the space $\mathrm{H}^{1/2}(\Gamma_i)$, $i = 1, 2$ is a subset (restrictions to the boundary Γ_i) of the trace space $\mathrm{H}^{1/2}(\partial\Omega)$ (of the Sobolev space $\mathrm{H}^1(\Omega)$). The space $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i = 1, 2$, consists of functions from $\mathrm{H}^{1/2}(\partial\Omega)$ vanishing on Γ_{3-i}, $i = 1, 2$, and $\left(\mathrm{H}^1(\Omega)\right)^*$ is the dual space of $\mathrm{H}^{1/2}_{00}(\Gamma_i)^d$, $i = 1, 2$, with the usual norms.

In this paper, we refer to steady-state heat conduction applications in (an)isotropic homogeneous media in the absence of heat sources. Consequently, the function $\mathrm{u}(\mathbf{x})$ denotes the temperature at a point $\mathbf{x} \in \Omega$ and satisfies the heat balance equation [1]

$$\mathcal{L}\mathrm{u}(\mathbf{x}) \equiv -\nabla \cdot \left(\mathbb{K}\,\nabla \mathrm{u}(\mathbf{x})\right) = 0\,, \quad \mathbf{x} \in \Omega\,. \tag{1}$$

Further, we let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector at $\partial\Omega$ and $\mathrm{q}(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$ defined by

$$\mathrm{q}(\mathbf{x}) \equiv -\mathbf{n}(\mathbf{x}) \cdot \left(\mathbb{K}\,\nabla \mathrm{u}(\mathbf{x})\right), \quad \mathbf{x} \in \partial\Omega\,. \tag{2}$$

If it is possible to measure both the temperature and the normal heat flux on a part of the boundary Γ, say Γ_1, then this leads to the mathematical formulation of the Cauchy problem consisting of the partial differential equation (1) and the boundary conditions

$$\mathrm{u}(\mathbf{x}) = \widetilde{\mathrm{u}}(\mathbf{x})\,, \qquad \mathrm{q}(\mathbf{x}) = \widetilde{\mathrm{q}}(\mathbf{x})\,, \qquad \mathbf{x} \in \Gamma_1\,, \tag{3}$$

where $\widetilde{\mathrm{u}} \in \mathrm{H}^{1/2}(\Gamma_1)$ and $\widetilde{\mathrm{q}} \in \left(\mathrm{H}^{1/2}_{00}(\Gamma_1)\right)^*$ are prescribed Dirichlet (temperature) and Neumann (normal heat flux) conditions, respectively. This problem consisting of (1) and (3), termed the Cauchy problem, is much more difficult to solve both analytically and numerically than direct problems, since its solution does not satisfy the general conditions of well-posedness [2].

Alternating Iterative Algorithms with Relaxation

We present two alternating iterative algorithms with relaxation, originally proposed by Jourhmane *et al.* [3], which aim at reducing the computational time of the alternating iterative algorithm of Kozlov *et al.* [4] for the simultaneous and stable reconstruction of the unknown temperature and normal heat flux on Γ_2, see also Marin [5, 6].

Alternating iterative algorithm with relaxation I:

Step 1. (i) If $k = 1$, specify an initial guess for the normal heat flux on Γ_2, i.e. $q^{(1)} \in \left(H^{1/2}_{00}(\Gamma_2)\right)^*$.
(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$\mathcal{L}u^{(2k-1)}(x) = 0\,, \quad x \in \Omega\,, \tag{4a}$$

$$q^{(2k-1)}(x) = \widetilde{q}(x)\,, \quad x \in \Gamma_1\,, \tag{4b}$$

$$u^{(2k-1)}(x) = u^{(2k-2)}(x)\,, \quad x \in \Gamma_2\,, \tag{4c}$$

to determine $u^{(2k-1)}(x)$, $x \in \Omega$, and $q^{(2k-1)}(x) \equiv -n(x) \cdot \left(\mathbb{K}\,\nabla u^{(2k-1)}(x)\right)$, $x \in \Gamma_2$.

Step 2. Update the unknown Neumann data on Γ_2 by setting

$$\xi^{(k)}(x) = \begin{cases} q^{(2k-1)}(x) & \text{for } k = 1 \\ \omega\, q^{(2k-1)}(x) + (1-\omega)\, \xi^{(k-1)}(x) & \text{for } k \geq 2 \end{cases}\,, \quad x \in \Gamma_2\,, \tag{5}$$

where $\omega \in (0, 2)$ is a fixed relaxation factor.
Having constructed the approximation $u^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$\mathcal{L}u^{(2k)}(x) = 0\,, \quad x \in \Omega\,, \tag{6a}$$

$$u^{(2k)}(x) = \widetilde{u}(x)\,, \quad x \in \Gamma_1\,, \tag{6b}$$

$$q^{(2k)}(x) = \xi^{(k)}(x)\,, \quad x \in \Gamma_2\,, \tag{6c}$$

is solved in order to determine $u^{(2k)}(x)$, $x \in \Omega$, and $u^{(2k)}(x)$, $x \in \Gamma_2$.

Step 3. Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Alternating iterative algorithm with relaxation II:

Step 1. (i) If $k = 1$, specify an initial guess for the boundary temperature on Γ_2, i.e. $u^{(1)} \in H^{1/2}(\Gamma_2)$.
(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$\mathcal{L}u^{(2k-1)}(x) = 0\,, \quad x \in \Omega\,, \tag{7a}$$

$$u^{(2k-1)}(x) = \widetilde{u}(x)\,, \quad x \in \Gamma_1\,, \tag{7b}$$

$$q^{(2k-1)}(x) = q^{(2k-2)}(x)\,, \quad x \in \Gamma_2\,, \tag{7c}$$

to determine $u^{(2k-1)}(x)$, $x \in \Omega$, and $u^{(2k-1)}(x)$, $x \in \Gamma_2$.

Step 2. Update the unknown Dirichlet data on Γ_2 by setting

$$\psi^{(k)}(x) = \begin{cases} u^{(2k-1)}(x) & \text{for } k = 1 \\ \omega\, u^{(2k-1)}(x) + (1-\omega)\, \psi^{(k-1)}(x) & \text{for } k \geq 2 \end{cases}\,, \quad x \in \Gamma_2\,, \tag{8}$$

where $\omega \in (0, 2)$ is a fixed relaxation factor.
Having constructed the approximation $u^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$\mathcal{L}u^{(2k)}(x) = 0\,, \quad x \in \Omega\,, \tag{9a}$$

$$q^{(2k)}(x) = \widetilde{q}(x)\,, \quad x \in \Gamma_1\,, \tag{9b}$$

$$u^{(2k)}(x) = \psi^{(k)}(x)\,, \quad x \in \Gamma_2\,, \tag{9c}$$

is solved in order to determine $u^{(2k)}(x)$, $x \in \Omega$, and $q^{(2k)}(x) \equiv -\mathbf{n}(x) \cdot \left(\mathbb{K}\,\nabla u^{(2k)}(x)\right)$, $x \in \Gamma_2$.

Step 3. Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

The convergence of the alternating iterative algorithm with relaxation II presented above can be recast in the following convergence theorem, with the mention that a similar result can also be obtained for the alternating iterative algorithm with relaxation I, see also [3, 5, 6]:

Theorem 1 *Let* $\widetilde{\mathrm{u}} \in \mathrm{H}^{1/2}(\Gamma_1)$ *and* $\widetilde{\mathrm{q}} \in \left(\mathrm{H}^{1/2}_{00}(\Gamma_1)\right)^*$, *and assume that the Cauchy problem* (1) *and* (3) *has a solution* $\mathrm{u} \in \mathrm{H}^1(\Omega)$. *Let* $\mathrm{u}^{(k)}$ *be the k-th approximate solution in the alternating procedure II described above. Then there exists a number* $1 < b \leq 2$ *such that when the relaxation parameter* ω *is chosen with* $1 \leq \omega \leq b$, *then*

$$\lim_{k\to\infty} \left\|\mathrm{u} - \mathrm{u}^{(k)}\right\|_{\mathrm{H}^1(\Omega)} = 0 \tag{10}$$

for any initial data element $\psi^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$.

The proof for this theorem in case of the proposed relaxation algorithms associated with the Cauchy problem for the steady-state (an)isotropic heat conduction is similar to that for the corresponding relaxation algorithms for the Cauchy problem in elasticity [7]. The proof given by Marin and Johansson [7] is based on the reformulation of the Cauchy problem (1) and (3) as a fixed point operator equation with a self-adjoint, injective, positive definite and non-expansive operator, while the scheme is shown to be a fixed point iteration for that equation. An alternative proof for the convergence result can also be found in [3]. As reported by Marin and Johansson [7] for Cauchy problems associated with the Navier-Lamé system of elasticity, it was also found for two-dimensional steady-state (an)isotropic heat conduction Cauchy problems that a relaxation factor $\omega > 2$ cannot be employed since the iterative process becomes divergent in such a situation.

The Method of Fundamental Solutions

The fundamental solution G of the heat balance equation (1) for two-dimensional steady heat conduction in (an)isotropic homogeneous media is given by, see e.g. Fairweather and Karageorghis [8]

$$G(\mathbf{x},\xi) = \begin{cases} \dfrac{1}{2\pi\sqrt{\det \mathbb{K}}} \ln \dfrac{1}{\sqrt{(\mathbf{x}-\xi)\cdot \mathbb{K}^{-1}(\mathbf{x}-\xi)}} & \text{for anisotropic media} \\ \dfrac{1}{2\pi} \ln \dfrac{1}{\sqrt{(\mathbf{x}-\xi)\cdot(\mathbf{x}-\xi)}} & \text{for isotropic media} \end{cases}, \quad \mathbf{x} \in \overline{\Omega}, \tag{11}$$

where $\xi \in \mathbb{R}^2 \setminus \overline{\Omega}$ is a singularity or source point. The main idea of the method of fundamental solutions (MFS) consists of the approximation of the temperature in the solution domain by a linear combination of fundamental solutions with respect to M singularities $\left\{\xi^{(j)}\right\}_{j=1}^{M} \subset \mathbb{R}^2 \setminus \overline{\Omega}$, in the form

$$\mathrm{u}(\mathbf{x}) \approx \mathrm{u}_M(\mathbf{c},\xi;\mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j\, G\left(\mathbf{x},\xi^{(j)}\right), \quad \mathbf{x} \in \overline{\Omega}, \tag{12}$$

where $\mathbf{c} = [\mathrm{c}_1,\ldots,\mathrm{c}_M]^{\mathsf{T}}$ and $\xi \in \mathbb{R}^{2M}$ is a vector containing the coordinates of the singularities $\left\{\xi^{(j)}\right\}_{j=1}^{M}$. From equations (2) and (11) it follows that the normal heat flux, through a curve defined by the outward unit normal vector $\mathbf{n}(\mathbf{x})$, can be approximated on the boundary, $\partial\Omega$, by

$$\mathrm{q}(\mathbf{x}) \approx \mathrm{q}_M(\mathbf{c},\xi;\mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j \left(-\mathbf{n}(\mathbf{x})\cdot \mathbb{K}\,\nabla_{\mathbf{x}} G\left(\mathbf{x},\xi^{(j)}\right)\right), \quad \mathbf{x} \in \partial\Omega. \tag{13}$$

Next, we select N_1 MFS collocation points $\left\{\mathbf{x}^{(i)}\right\}_{i=1}^{N_1}$ on the boundary Γ_1 and N_2 MFS collocation points $\left\{\mathbf{x}^{(N_1+i)}\right\}_{i=1}^{N_2}$ on the boundary Γ_2, such that the total number of MFS collocation points used to discretise the boundary $\partial\Omega$ of the solution domain Ω is given by $N = N_1 + N_2$. Further, we define the vectors $\mathbf{U}^{(j)} \in \mathbb{R}^{N_j}$ and $\mathbf{Q}^{(j)} \in \mathbb{R}^{N_j}$, $j = 1,2$, containing the values of the temperature and normal heat flux, respectively, at the collocation points on the boundary Γ_j, $j = 1,2$. We also denote by $\mathbf{A}^{(j)} \in \mathbb{R}^{N_j\times M}$ and $\mathbf{B}^{(j)} \in \mathbb{R}^{N_j\times M}$, $j = 1,2$, the matrices that determine the MFS approximation of the temperature and normal heat flux on Γ_j, $j = 1,2$, respectively. Using these notations, the MFS discretisation of the Cauchy boundary data (3) and the MFS approximation of the unknown boundary conditions on Γ_2 are given by the following relations with $j = 1$ and $j = 2$, respectively,

$$\mathbf{A}^{(j)}\,\mathbf{c} = \mathbf{U}^{(j)}, \qquad \mathbf{B}^{(j)}\,\mathbf{c} = \mathbf{Q}^{(j)}. \tag{14}$$

Note that all components of the matrices $\mathbf{A}^{(j)}$, $\mathbf{B}^{(j)}$, $j = 1, 2$, and the vectors $\mathbf{U}^{(1)}$, $\mathbf{Q}^{(1)}$ are known.

At each step of the alternating iterative algorithms with relaxation I and II, two direct mixed well-posed problems are solved using the MFS. The general form of the MFS systems of linear algebraic equations associated with these direct problems may be recast as

$$\mathbf{A}_k^{(\mathrm{I})}\,\mathbf{c}_k^{(\mathrm{I})} = \mathbf{f}_k^{(\mathrm{I})} \quad \text{and} \quad \mathbf{A}_k^{(\mathrm{II})}\,\mathbf{c}_k^{(\mathrm{II})} = \mathbf{f}_k^{(\mathrm{II})}\,, \quad k \geq 1\,, \tag{15}$$

where the superscripts (I) and (II) refer to the alternating iterative algorithms with relaxation I and II, respectively, while the subscript k represents the iteration number. Since the direct problems (4) and (9), and (6) and (7) are of the same kind, but with different right-hand sides, we obtain:

(i) For problems (4) and (9)

$$\mathbf{A}_{2k-1}^{(\mathrm{I})} = \mathbf{A}_{2k}^{(\mathrm{II})} = \left[\mathbf{B}^{(1)} \quad \mathbf{A}^{(2)}\right], \quad \mathbf{f}_{2k-1}^{(\mathrm{I})} = \begin{bmatrix} \widetilde{\mathbf{Q}}^{(1)} \\ \mathbf{U}_{2k-1}^{(2)} \end{bmatrix}, \quad \mathbf{f}_{2k}^{(\mathrm{II})} = \begin{bmatrix} \widetilde{\mathbf{Q}}^{(1)} \\ \mathbf{\Psi}_k^{(2)} \end{bmatrix}, \quad k \geq 1. \tag{16}$$

(ii) For problems (6) and (7)

$$\mathbf{A}_{2k}^{(\mathrm{I})} = \mathbf{A}_{2k-1}^{(\mathrm{II})} = \left[\mathbf{A}^{(1)} \quad \mathbf{B}^{(2)}\right], \quad \mathbf{f}_{2k}^{(\mathrm{I})} = \begin{bmatrix} \widetilde{\mathbf{U}}^{(1)} \\ \mathbf{\Xi}_k^{(2)} \end{bmatrix}, \quad \mathbf{f}_{2k-1}^{(\mathrm{II})} = \begin{bmatrix} \widetilde{\mathbf{U}}^{(1)} \\ \mathbf{Q}_{2k-1}^{(2)} \end{bmatrix}, \quad k \geq 1. \tag{17}$$

In order to uniquely determine the solutions $\mathbf{c}_k^{(\mathrm{I})} \in \mathbb{R}^M$ and $\mathbf{c}_k^{(\mathrm{II})} \in \mathbb{R}^M$ to the MFS systems of N linear algebraic equations with M unknowns given in (15), the numbers of MFS boundary collocation points and singularities must satisfy the inequality $M \leq N$. However, these MFS systems cannot be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution in the case of noisy Cauchy data on Γ_1. Therefore, the MFS systems of N linear algebraic equations with M unknowns given in (15) are solved, in a stable manner, by using the Tikhonov regularization method [9], while the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion [10].

Numerical Results

Example. In the following, we solve the Cauchy problem (1) and (3) for an anisotropic solid characterised by the thermal conductivity tensor $\mathrm{K}_{11} = 1.0$, $\mathrm{K}_{12} = \mathrm{K}_{21} = 0.5$, $\mathrm{K}_{22} = 1.0$ and occupying the disk $\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 \,\middle|\, \rho(\mathbf{x}) < 1\right\}$, where $\rho(\mathbf{x}) = \sqrt{\mathrm{x}_1^2 + \mathrm{x}_2^2}$ is the radial polar coordinate of $\mathbf{x}$. The following analytical solutions for the temperature and normal heat flux are assumed:

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x}) = \mathrm{x}_1^2 - 4\mathrm{x}_1\mathrm{x}_2 + \mathrm{x}_2^2\,, \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \overline{\Omega}\,, \tag{18a}$$

$$\mathrm{q}^{(\mathrm{an})}(\mathbf{x}) = 3\left[\mathrm{x}_2\,\mathrm{n}_1(\mathbf{x}) + \mathrm{x}_1\,\mathrm{n}_2(\mathbf{x})\right]\,, \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \partial\Omega\,, \tag{18b}$$

respectively. Here $\Gamma_1 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, 0 \leq \theta(\mathbf{x}) \leq 3\pi/2\right\}$ and $\Gamma_2 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, 3\pi/2 < \theta(\mathbf{x}) < 2\pi\right\}$, where $\theta(\mathbf{x})$ is the angular polar coordinate of $\mathbf{x}$.

In this study, we have adopted the so-called static approach associated with the MFS singularities, in the sense that these singularities are preassigned and kept fixed throughout the solution process. More precisely, the MFS singularities have been taken on the pseudo-boundary $\partial\Omega_{\mathrm{S}}$ at the distance d_{S} from the physical boundary $\partial\Omega$. The inverse problem considered herein was solved using a uniform distribution of both the MFS boundary collocation points $\left\{\mathbf{x}^{(i)}\right\}_{i=1}^{N}$ and the singularities $\left\{\boldsymbol{\xi}^{(j)}\right\}_{j=1}^{M}$, such that $N_1/3 = N_2 = N/4 = 20$, $M = N/2$ and $d_{\mathrm{S}} = 3.0$.

Convergence of the Algorithms. If L_i collocation points, $\left\{\mathbf{z}^{(\ell)}\right\}_{\ell=1}^{L_i}$, are considered on the boundary $\Gamma_i \subset \partial\Omega$ then the *root mean square error* (RMS error) associated with the real valued function $f(\cdot) : \Gamma_i \longrightarrow \mathbb{R}$ on Γ_i is defined by

$$\mathrm{RMS}_{\Gamma_i}(f) = \sqrt{\frac{1}{L_i}\sum_{\ell=1}^{L_i} f\left(\mathbf{z}^{(\ell)}\right)^2}\,, \tag{19}$$

In order to analyse the accuracy, convergence and stability of the proposed alternating iterative algorithms with relaxation, for $k \geq 1$, we evaluate the following accuracy errors corresponding to the temperature and normal heat flux on Γ_2, which are defined as *relative RMS errors*, i.e.

$$e_u(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_2}\left(u^{(2k)} - u^{(an)}\right)}{\mathrm{RMS}_{\Gamma_2}\left(u^{(an)}\right)} & \text{for the alternating iterative algorithm with relaxation I} \\ \dfrac{\mathrm{RMS}_{\Gamma_2}\left(u^{(2k-1)} - u^{(an)}\right)}{\mathrm{RMS}_{\Gamma_2}\left(u^{(an)}\right)} & \text{for the alternating iterative algorithm with relaxation II} \end{cases} \tag{20a}$$

and

$$e_q(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_2}\left(q^{(2k-1)} - q^{(an)}\right)}{\mathrm{RMS}_{\Gamma_2}\left(q^{(an)}\right)} & \text{for the alternating iterative algorithm with relaxation I} \\ \dfrac{\mathrm{RMS}_{\Gamma_2}\left(q^{(2k)} - q^{(an)}\right)}{\mathrm{RMS}_{\Gamma_2}\left(q^{(an)}\right)} & \text{for the alternating iterative algorithm with relaxation II}. \end{cases} \tag{20b}$$

Here $u^{(2k)}$ $\left(u^{(2k-1)}\right)$ and $q^{(2k-1)}$ $\left(q^{(2k)}\right)$ are the temperature and normal heat flux on the boundary Γ_2 retrieved after k iterations using the alternating iterative algorithm with relaxation I (II), respectively.

Although not presented, it is reported that, for all admissible values of the relaxation parameter, $\omega \in (0, 2)$, both accuracy errors e_u and e_q keep decreasing until a specific number of iterations, after which the convergence rate of the aforementioned errors becomes very slow so that they reach a plateau. As expected, for each value of ω, $e_u(k) < e_q(k)$ for all $k \geq 1$; also, the larger the parameter ω, the lower the number of iterations and, consequently, computational time are required for obtaining accurate numerical results for both the the temperature and the normal heat flux on Γ_1.

Regularizing Stopping Criterion. To simulate the inherent inaccuracies in the measured data on Γ_1, we assume that various levels of Gaussian random noise, p_u and p_q, have been added into the exact temperature $u|_{\Gamma_1} = \widetilde{u}$ and normal heat flux $q|_{\Gamma_1} = \widetilde{q}$ data, respectively, so that the following perturbed temperature and normal heat flux are available on Γ_1:

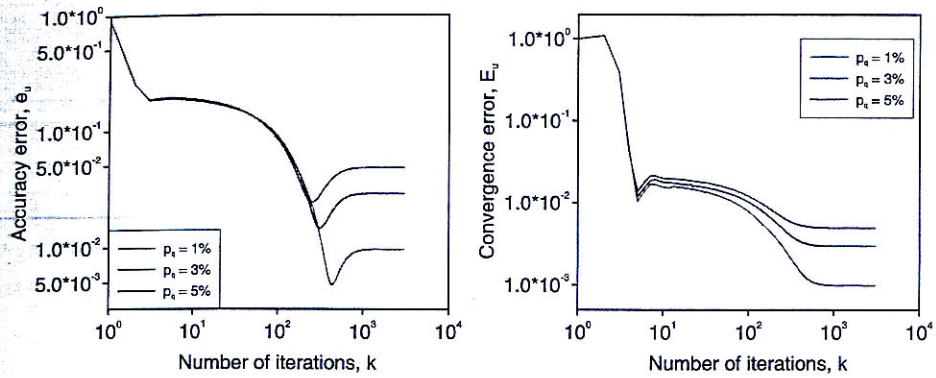
$$\widetilde{u}^\epsilon \in L^2(\Gamma_1): \ \left\|u^{(an)}\big|_{\Gamma_1} - \widetilde{u}^\epsilon\right\|_{L^2(\Gamma_1)} = \epsilon \quad \text{and} \quad \widetilde{q}^\epsilon \in L^2(\Gamma_1): \ \left\|q^{(an)}\big|_{\Gamma_1} - \widetilde{q}^\epsilon\right\|_{L^2(\Gamma_1)} = \epsilon\,. \tag{21}$$

Fig. 1(a) presents, on a logarithmic scale, the accuracy error e_u as a function of the number of iterations, k, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and $p_q \in \{1\%, 3\%, 5\%\}$. From this figure it can be seen that, for each fixed value of p_q, the errors in predicting the temperature on the under-specified boundary Γ_2 decrease up to a certain iteration number and after that they start increasing. Although not illustrated, it is important to mention that the accuracy error e_q has a similar behaviour. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion must be used in order to terminate the iterative process at the point where the errors in the numerical solutions start increasing.

To define the stopping criterion required for regularizing/stabilizing the iterative methods analysed in this paper, after each iteration, k, we evaluate the following convergence error which is associated with the temperature on the over-specified boundary, Γ_1, namely

$$E_u(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_1}\left(u^{(2k-1)} - \widetilde{u}^\epsilon\right)}{\mathrm{RMS}_{\Gamma_1}\left(\widetilde{u}^\epsilon\right)} & \text{for the alternating iterative algorithm with relaxation I} \\ \dfrac{\mathrm{RMS}_{\Gamma_1}\left(u^{(2k)} - \widetilde{u}^\epsilon\right)}{\mathrm{RMS}_{\Gamma_1}\left(\widetilde{u}^\epsilon\right)} & \text{for the alternating iterative algorithm with relaxation II}. \end{cases} \tag{22}$$

Here $u^{(2k-1)}$ $\left(u^{(2k)}\right)$ is the temperature on the boundary Γ_1, retrieved numerically after k iterations by solving the well-posed mixed direct boundary value problem (4a)–(4c) [(9a)–(9c)], in the case of the



(a) Accuracy error e_u: $p_q \in \{1\%, 3\%, 5\%\}$, algorithm I (b) Convergence error E_u: $p_q \in \{1\%, 3\%, 5\%\}$, algorithm I

Figure 1: (a) The accuracy errors e_u, and (b) the convergence error, E_u, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into $q|_{\Gamma_1}$.

alternating iterative algorithm I (II), while $\tilde{u}^\epsilon$ is the perturbed Dirichlet data (boundary temperature) on the over-specified boundary Γ_1, as given by (21). This error E_u should tend to zero as the sequences $\{u^{(2k-1)}\}_{k\geq 1}$ and $\{u^{(2k)}\}_{k\geq 1}$ tend to the analytical solution, $u^{(an)}$, in the space $H^1(\Omega)$ and hence it is expected to provide an appropriate stopping criterion. Indeed, if we investigate the error E_u obtained at each iteration using the alternating iterative algorithm I, $\omega = 1.50$ and $p_q \in \{1\%, 3\%, 5\%\}$, we obtain the curves graphically represented in Fig. 1(b). By comparing Figs. 1(a) and (b), it can be noticed that both the convergence error E_u and the accuracy error e_u attain their corresponding minimum at around the same number of iterations. Therefore, for noisy Cauchy data a natural stopping criterion terminates the MFS iterative algorithms with relaxation I and II at the optimal number of iterations, k_{opt}, given by:

$$k_{opt}: \quad E_u(k_{opt}) = \min_{k\geq 1} E_u(k)\,. \tag{23}$$

Although not illustrated, it is reported that similar results and conclusions have been obtained for all admissible values of ω, as well as the MFS-based iterative algorithm with relaxation II.

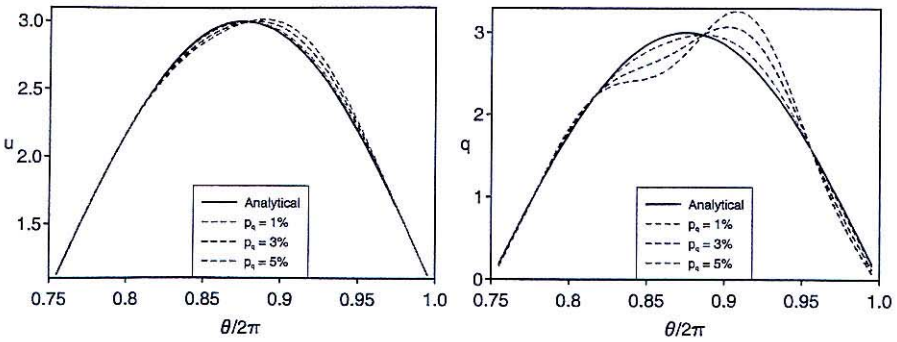
Stability of the Algorithms. Based on the stopping criterion (23), the numerical results for the temperature and normal heat flux on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and $p_q \in \{1\%, 3\%, 5\%\}$, and their corresponding analytical values are presented in Figs. 2(a) and (b), respectively. It can be seen from these figures that the numerical solutions for both the temperature and the normal heat flux on Γ_2 are stable approximation to their corresponding exact solutions, free of unbounded and rapid oscillations, and they converge to the exact solutions as $p_q \longrightarrow 0$.

The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the CPU time, obtained using the alternating iterative algorithm I, the stopping criterion (23), various levels of noise added into the Dirichlet and Neumann data on Γ_1 and various values of the relaxation parameter, $\omega \in (0, 2)$, are presented in Table 1.

In order to assess the performance of the alternating iterative algorithm I with under-, no and over-relaxation, we exemplify by considering $p_u = 1\%$ and $p_q = 0\%$. In this case, the CPU times needed for the alternating iterative algorithm I with $\omega = 0.50$ (under-relaxation), $\omega = 1.00$ (no relaxation) and $\omega = 1.50$ (over-relaxation) to reach the numerical solutions for the temperature and normal heat flux on Γ_2 were found to be 5785.39, 3455.76 and 1674.59 s, respectively, while the corresponding values for the optimal iteration number required, k_{opt}, were found to be 4136, 2755 and

ω	p_u	p_q	k_{opt}	$e_u(k_{opt})$	$e_q(k_{opt})$	CPU time [s]
0.20	1%	0%	4958	0.12065×10^{-1}	0.51310×10^{-1}	6845.51
	3%	0%	4716	0.36192×10^{-1}	0.15391×10^{0}	6029.39
	5%	0%	4483	0.60320×10^{-1}	0.25652×10^{0}	5228.04
0.50	1%	0%	4136	0.12065×10^{-1}	0.51310×10^{-1}	5785.39
	3%	0%	3934	0.36192×10^{-1}	0.15391×10^{0}	5306.01
	5%	0%	3740	0.60320×10^{-1}	0.25652×10^{0}	4327.90
1.00	1%	0%	2755	0.12065×10^{-1}	0.51310×10^{-1}	3455.76
	3%	0%	2620	0.36192×10^{-1}	0.15391×10^{0}	3269.54
	5%	0%	2490	0.60320×10^{-1}	0.25652×10^{0}	2868.35
1.50	1%	0%	1367	0.12065×10^{-1}	0.51310×10^{-1}	1674.59
	3%	0%	1299	0.36192×10^{-1}	0.15391×10^{0}	1560.15
	5%	0%	1234	0.60320×10^{-1}	0.25652×10^{0}	1396.48
1.80	1%	0%	508	0.12065×10^{-1}	0.51310×10^{-1}	603.45
	3%	0%	483	0.36192×10^{-1}	0.15391×10^{0}	572.96
	5%	0%	458	0.60320×10^{-1}	0.25652×10^{0}	523.65
0.20	0%	1%	1584	0.48522×10^{-2}	0.26140×10^{-1}	1883.54
	0%	3%	1147	0.14558×10^{-1}	0.78556×10^{-1}	1302.14
	0%	5%	905	0.24264×10^{-1}	0.13092×10^{0}	1088.95
0.50	0%	1%	1323	0.48522×10^{-2}	0.26164×10^{-1}	1549.07
	0%	3%	960	0.14558×10^{-1}	0.78571×10^{-1}	1284.92
	0%	5%	763	0.24264×10^{-1}	0.13091×10^{0}	871.51
1.00	0%	1%	880	0.48522×10^{-2}	0.26163×10^{-1}	1106.43
	0%	3%	638	0.14558×10^{-1}	0.78511×10^{-1}	733.87
	0%	5%	506	0.24264×10^{-1}	0.13097×10^{0}	661.15
1.50	0%	1%	435	0.48522×10^{-2}	0.26152×10^{-1}	585.96
	0%	3%	310	0.14558×10^{-1}	0.78789×10^{-1}	445.68
	0%	5%	249	0.24264×10^{-1}	0.13084×10^{0}	355.50
1.80	0%	1%	162	0.48522×10^{-2}	0.26349×10^{-1}	187.09
	0%	3%	116	0.14558×10^{-1}	0.79429×10^{-1}	156.56
	0%	5%	92	0.24264×10^{-1}	0.13120×10^{0}	115.96

Table 1: The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, and the computational time, obtained using the alternating iterative algorithm I, the regularizing stopping criterion (23), various amounts of noise added into $u|_{\Gamma_1}$ or $q|_{\Gamma_1}$ and various values for the relaxation parameter, ω.



(a) Example 1: Temperatures on Γ_2 (b) Example 1: Normal heat fluxes on Γ_2

Figure 2: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into $q|_{\Gamma_1}$.

1367, respectively. This means that, to attain the numerical solutions for the unknown Dirichlet and Neumann data on Γ_2, the alternating iterative algorithm I with over-relaxation ($\omega = 1.50$) requires a reduction in the number of iterations performed and CPU time by approximately 50% and 67% with respect to those corresponding to the standard iterative algorithm I as proposed by Kozlov *et al.* [4], i.e. without relaxation ($\omega = 1.00$), and the alternating iterative algorithm I with under-relaxation ($\omega = 0.50$), respectively.

Conclusions

We proposed two algorithms involving the relaxation of either the given boundary temperature or the prescribed normal heat flux for the iterative algorithm of Kozlov *et al.* [4] applied to two-dimensional steady-state (an)isotropic heat conduction Cauchy problems. The two well-posed and direct problems corresponding to each iteration were solved using the MFS and the Tikhonov regularization method, while the optimal value of the regularization parameter was selected via the GCV criterion. An efficient regularizing stopping criterion was also presented. The numerical results obtained showed the numerical stability, convergence and computational efficiency of the proposed relaxation procedures.

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