A VARIATIONAL FORMULATION FOR CONSTITUTIVE LAWS DESCRIBED BY BIPOTENTIALS

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Abstract. Inspired by the algorithm for solving the discretisation in time of the evolution problem for an implicit standard material, presented in [1], we propose a variational formulation in terms of bipotentials.

1. Introduction

In the paper [1] Berga and de Saxcé propose a bipotential for the constitutive law of a soil and further they proceed with the variational formulation of this model.

We are interested in the precise formulation of the model, especially we want to understand from a mathematical viewpoint the recipe proposed in [1] for using bipotentials in order to get a variational formulation of their model. We regard this as the first step towards the establishment of a general variational theory of bipotentials.

The following paragraph, extracted from [1] page 414, is revealing for two reasons: (a) the understanding of the motivation for introducing the bifunctional in order to adapt the Uzawa algorithm for implicit constitutive laws; (b) the imprecision concerning the understanding of the proposed new algorithm, related to the fact that, as we shall see, the simultaneous minimization of the bifunctional is not in fact how the algorithm works.

"*One of the advantages of the new formulation is to extend the classical Calculus of Variations to non associated constitutive laws. In the theoretical frame of the Implicit Standard Materials, a new functional, called bifunctional, is introduced, depending on both the displacement and stress field. The exact solution*

1991 *Mathematics Subject Classification.* Primary: 74A20, 49J40; Secondary: 26B25.

Key words and phrases. Bipotentials theory, Variational principles, Nonassociated constitutive laws.

The first author is partially supported by the grant "Continuous modeling of advanced materials in virtual fabrication" COMOD PCCE - ID 100. All authors acknowledge the support from the European Associated Laboratory "Math Mode" associating the Laboratoire de Mathématiques de l'Université Paris-Sud (UMR 8628) and the "Simion Stoilow" Institute of Mathematics of the Romanian Academy.

of the Boundary Value Problem corresponds to the simultaneous minimization of the bifunctional, firstly with respect to kinematically admissible displacement fields, when the stress field is equal with the exact one, and secondly with respect to statically admissible stress fields, when the displacement field is the exact one. The two minimization problems are the direct extension of the dual variational principles of displacements and stresses."

The notion of bipotential (definition 3.1) has been introduced in [20], in order to formulate a large family of non associated constitutive laws in terms of convex analysis. The basic idea is explained further in few words. In Mechanics the associate constitutive laws are simply relations $y \in \partial\phi(x)$, with $\phi : X \to \mathbb{R} \cup \{+\infty\}$ a convex and lower semicontinuous function. By Fenchel inequality such a relation is equivalent with $\phi(x) + \phi^*(y) = \langle x, y\rangle$, where ϕ^* is the Fenchel conjugate of ϕ. It has been noticed that often in the mathematical study of problems related to associated constitutive laws enters not the function ϕ, but the expression

$$b(x, y) = \phi(x) + \phi^*(y)$$

which we call "separable bipotential". The idea is then to use as a basic notion the one of bipotential $b : X \times Y \to \mathbb{R} \cup \{+\infty\}$, which is convex and lsc in each argument and satisfies a generalization of the Fenchel inequality. To non associated constitutive laws thus corresponds bipotentials which are not separable.

There are many such laws which can be studied with the help of bipotentials, as witnessed by the papers listed further. In many of these papers bipotentials are used for numerical purposes and several ad hoc algorithms have been suggested and exploited for applications. Here is a partial list of constitutive laws which have been described by bipotentials: non-associated Drücker-Prager [18] and Cam-Clay models [17] in soil mechanics, cyclic Plasticity ([16],[3]) and Viscoplasticity [11] of metals with non linear kinematical hardening rule, Lemaitre's damage law [2], the coaxial laws ([19],[22]), the Coulomb's friction law [20], [16], [4], [9], [10], [12], [18], [21], [13], [7]. A complete survey can be found in [19].

Later we started in [5] [6] [7] a mathematical study of bipotentials and their relation with convex analysis. This paper is another contribution along this subject, concerning mathematically sound variational formulations and algorithms for numerically solving the quasistatic evolution problem for constitutive laws of implicit standard materials. For another paper which contains a variational formulation via bipotentials for the particular case of separated bipotentials, see [14].

2. Notations and prerequisites from convex analysis

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot\rangle : X \times Y \to \mathbb{R}$. We shall suppose that X, Y have topologies compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y\rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y\rangle$, for some $x \in X$). We use the notations:

- $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$;
- the domain of a function $\phi : X \to \bar{\mathbb{R}}$ is $dom\, \phi = \{x \in X \,:\, \phi(x) \in \mathbb{R}\}$;
- $\Gamma_0(X) = \left\{\phi : X \to \bar{\mathbb{R}} \,:\, \phi \text{ is lsc and } dom\phi \neq \emptyset\right\}$;

- for any convex and closed set $A \subset X$, its indicator function, Ψ_A, is defined by

$$\Psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}$$

- the subgradient of a function $\phi : X \to \bar{\mathbb{R}}$ at a point $x \in X$ is the (possibly empty) set:

$$\partial\phi(x) = \{u \in Y \mid \forall z \in X \ \langle z - x, u\rangle \leq \phi(z) - \phi(x)\} \ .$$

- the inf-convolution of two functions $\phi, \psi \in \Gamma_0(X)$ is the function $\phi\square\psi \in \Gamma_0(X)$ defined by: for any $x \in X$

$$\phi\square\psi(x) \ = \ \inf\{\phi(u) + \psi(v) : u + v = x\}$$

3. Bipotentials and syncs

Definition 3.1. A bipotential is a function $b : X \times Y \to \bar{\mathbb{R}}$, with the properties:

(a) for any $x \in X$, if $dom\, b(x, \cdot) \neq \emptyset$ then $b(x, \cdot) \in \Gamma_0(X)$; for any $y \in Y$, if $dom\, b(\cdot, y) \neq \emptyset$ then $b(\cdot, y) \in \Gamma_0(Y)$;
(b) for any $x \in X, y \in Y$ we have $b(x, y) \geq \langle x, y\rangle$;
(c) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y\rangle \ . \tag{1}$$

The graph of b is

$$M(b) \ = \ \{(x, y) \in X \times Y \ \mid \ b(x, y) = \langle x, y\rangle\} \ . \tag{2}$$

Bipotentials are related to syncronised convex functions, defined further.

Definition 3.2. A sync (syncronised convex function) is a function $c : X \times Y[0, +\infty]$ with the properties:

(a) for any $x \in X$, if $dom\, c(x, \cdot) \neq \emptyset$ then $c(x, \cdot) \in \Gamma_0(X)$; for any $y \in Y$, if $dom\, c(\cdot, y) \neq \emptyset$ then $c(\cdot, y) \in \Gamma_0(Y)$;
(b) for any $x \in X$, if $dom\, c(x, \cdot) \neq \emptyset$ and the minimum $\min\{c(x, y) : y \in Y\}$ exists then this minimum equals 0; for any $y \in X$, if $dom\, c(\cdot, y) \neq \emptyset$ and the minimum $\min\{c(x, y) : x \in X\}$ exists then this minimum equals 0.

Proposition 1. *A function $b : X \times Y \to \bar{\mathbb{R}}$ is a bipotential if and only if the function $c : X \times Y \to \bar{\mathbb{R}}$, $c(x, y) = b(x, y) - \langle x, y\rangle$ is a sync.*

Remark 1. The string of equivalences (1) justifies the name "syncronised convex function", as it expresses the fact that critical points of functions $c(x, \cdot)$ are related with critical points of functions $c(\cdot, y)$.

With the notations from proposition 1, we have $M(b) = c^{-1}(0)$. Also, for any $x \in X$ and $y \in Y$, property (a) definition 3.2 of syncs is equivalent with:

$$epi(c) \cap \{x\} \times Y \times \mathbb{R} \text{ and } epi(c) \cap X \times \{y\} \times \mathbb{R}$$

are closed convex sets, where $epi(c)$ is the epigraph of c:

$$epi(c) = \{(x, y, r) \in X \times Y \times \mathbb{R} : c(x, y) \leq r\}$$

An interesting fact is that duality products do not enter in the definition of syncs. As an application, let (X, Y) be a pair of spaces, $\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle'$ be two duality products, defined on $X \times Y$, and $c : X \times Y \to [0, +\infty]$ be a sync. We define the applications:

$$b\,,\, b' : X \times Y \to \mathbb{R} \cup \{+\infty\} \quad b(x,y) = c(x,y) + \langle x,y \rangle\,,\, b'(x,y) = c(x,y) + \langle x,y \rangle'$$

Then b is a bipotential with respect to $\langle \cdot, \cdot \rangle$ and b' is a bipotential with respect to $\langle \cdot, \cdot \rangle'$. As a corollary, if we have a bipotential b with respect to the duality product $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ is another duality product, then the application b' defined by

$$b'(x,y) = b(x,y) - \langle x,y \rangle \ + \langle x,y \rangle'$$

is a bipotential with respect to the duality product $\langle \cdot, \cdot \rangle'$ and $M(b) = M(b')$ (they describe the same law). More generally, we have the following proposition concerning transformations of syncs.

Proposition 2. *Let $(X, Y, \langle \cdot, \cdot \rangle)$, $(X', Y', \langle \cdot, \cdot \rangle')$ be two pairs of spaces with their respective duality products, $T : X \to X'$ and $L : Y \to Y'$ be two linear, bijective, continuous transformations, $\alpha > 0$ and $c' : X' \times Y' \to [0, +\infty]$ be a sync. Then the function*

$$c : X \times Y \to [0, +\infty] \quad , \quad c(x,y) \ = \ \alpha\, c'(Tx, Ly)$$

is a sync and $c^{-1}(0) = c'^{-1}(0)$.

Proof. The application c' is a sync, therefore it satisfies conditions (a), (b) from definition 3.2. It is straightforward that c is convex and lsc in each argument, therefore condition (a) definition 3.2 is a consequence of the same condition for c'. Also, because T and L are bijective, condition (b) for the application c follows from the same condition for c'. □

The following is definition 3.1 [5].

Definition 3.3. A non empty set $M \subset X \times Y$ is a BB-graph (bi-convex and bi-closed) if for any $x \in X$ and $y \in Y$ the sections

$$M(x) \ = \ \{y \in Y \mid (x,y) \in M\}$$
$$M^*(y) \ = \ \{x \in X \mid (x,y) \in M\}$$

are convex and closed.

For any BB-graph M the indicator function Ψ_M is obviously a sync. To this sync corresponds the bipotential

$$b_\infty(x,y) = \langle x,y \rangle + \Psi_M(x,y).$$

In particular, this shows that to a BB-graph we may associate more than one bipotential. Indeed, if M is maximal cyclically monotone then it is the graph of a separable bipotential, but also the graph of the bipotential associated to the sync Ψ_M (that is a bipotential of the form b_∞). Therefore maximal cyclically monotone graphs admit at least two distinct bipotentials.

4. Implicit standard materials described by bipotentials

In the mechanics of standard materials, the evolution problem is generally given by a set of equations, inequations, boundary and initial conditions. They can be structured in three groups: kinematical equations, equilibrium equations and the constitutive law modeling the material behavior.

4.1. **Notations.** The configuration of the body is represented by Ω, an open, bounded set with piecewise smooth boundary $\partial\Omega$.

We denote by n the dimension of the configuration space ($n = 1, 2$ or 3), thus $\Omega \subset \mathbb{R}^n$.

The boundary decomposes in two disjoint parts: on $\partial_0\Omega$ displacements are imposed, while given surface forces act on the remaining part of the boundary denoted by $\partial_1\Omega$. The closure of Ω is denoted by $\bar{\Omega}$.

The following quantities are considered.

- u is the displacement of the body with respect to the configuration Ω,
- $\varepsilon = D(u) = \dfrac{1}{2}\left(\nabla u + \nabla u^T\right)$ is the associated strain. The trace of the strain is denoted by $e_m = \dfrac{1}{n}\, tr\,\varepsilon$ and the strain deviator is
$$e = \varepsilon - e_m I_n$$
- The strain ε decomposes additively into elastic and plastic strains
$$\varepsilon = \varepsilon^e + \varepsilon^p$$
The traces of elastic strain ε^e and plastic strain ε^p are denoted respectively by e_m^e, e_m^p, and their deviatoric parts are e^e, e^p respectively,
- The stress field is denoted by σ, its trace is the hydrostatic pressure $s_m = tr\,\sigma$ and s denotes the stress deviator.
- S is the elasticity tensor modulus.
- The density of volumic forces is f_v; on $\partial_1\Omega$ act the surface forces with density f_s. The class of stress fields σ which satisfy the equilibrium equations:
$$div\,\sigma + f_v = 0 \text{ in } \Omega \quad , \quad \sigma \cdot n = f_s \text{ on } \partial_1\Omega$$
is denoted by $SA(f_v, f_s)$.
- The imposed boundary displacements on $\partial_0\Omega$ are denoted by $\bar{u}$. In fact, it is useful for further computations to consider the imposed boundary displacement $\bar{u}$ to be defined over all $\bar{\Omega}$. The class of displacements u, such that $u - \bar{u} = 0$ on $\partial\Omega$ (possibly in the sense of trace) is denoted by $CA(\bar{u})$ and called the class of displacements which are kinematically admissible with respect to $\bar{u}$.

Let $Sym(n)$ be the space of $n \times n$ real symmetric matrices and $Sym_0(n) \subset Sym(n)$ the subspace of real symmetric matrices with null trace. The decomposition of a real symmetric matrix into hydrostatic and deviatoric parts can be expressed by the linear bijective transformations:
$$T_1 : Sym(n) \to \mathbb{R} \times Sym_0(n) \quad , \quad T_1(\varepsilon) = \left(\frac{1}{n}\, tr\,\varepsilon, \varepsilon - \frac{1}{n}(tr\,\varepsilon) I_n\right)$$
$$T_2 : Sym(n) \to \mathbb{R} \times Sym_0(n) \quad , \quad T_2(\sigma) = \left(tr\,\sigma, \sigma - \frac{1}{n}(tr\,\sigma) I_n\right)$$
With the notations previously made, for any strain value $\varepsilon \in Sym(n)$, or for any stress value $\sigma \in Sym(n)$, the decompositions in hydrostatic and deviatoric parts are:
$$T_1(\varepsilon) = (e_m, e) \quad , \quad T_2(\sigma) = (s_m, s)$$
(In order to keep track of physical dimensions, we should introduce two spaces $Sym(n)$, one for strains and the other for stresses, or introduce units of measure,

but we feel that such notations are only making the presentation unnecessary complicated.)

We shall consider the following duality products:

$$\langle \cdot , \cdot \rangle : Sym(n) \times Sym(n) \rightarrow \mathbb{R} \quad , \quad \langle \varepsilon, \sigma \rangle = \, tr\,(\varepsilon \sigma)$$

$$\langle \cdot , \cdot \rangle' : (\mathbb{R} \times Sym_0(n)) \times (\mathbb{R} \times Sym_0(n)) \rightarrow \mathbb{R} \quad , \quad \langle (e_m, e), (s_m, s) \rangle' = e_m s_m + \langle e, s \rangle$$

Remark that the first duality product is the one entering in the formulation of the dissipation (as an integral over the body configuration Ω of $\langle \dot{\varepsilon^p}, \sigma$. The second duality product will be used for the plastic bipotential, see later for the example of the Berga & de Saxcé bipotential for the non-associative Drücker-Prager law. The relation between these dualities is:

$$\langle \varepsilon, \sigma \rangle = \langle T_1(\varepsilon), T_2(\sigma) \rangle'$$

therefore (by passing to associated syncs and back) we can easily transform bipotentials expressed in coordinates (ε, σ) into bipotentials expressed in coordinates $((e_m, e), (s_m, s))$.

The kinematical equations are:

$$\varepsilon = \frac{1}{2} \left(\nabla u + \nabla u^T \right) \quad , \quad u \in CA(\bar{u}) \tag{3}$$

The equilibrium equations are:

$$\sigma \in SA(f_v, f_s) \tag{4}$$

The constitutive equations (besides the additive decomposition of the strain into elastic and plastic parts) are expressed with two bipotentials: the elastic and the plastic bipotential respectively.

The elastic bipotential is defined by the elasticity tensor modulus and it has the form:

$$b_e(\varepsilon^e, \sigma) = \frac{1}{2} \langle \varepsilon^e, S \varepsilon^e \rangle + \frac{1}{2} \langle S^{-1} \sigma, \sigma \rangle \tag{5}$$

The elastic bipotential is defined over pairs of dual variables (elastic strain, stress). It is a separable bipotential, expresses as the sum of the (density of) the elastic energy and it's dual. Moreover, this bipotential is quadratic in each variable.

The plastic bipotential

$$b_p = b_p(\dot{\varepsilon}^p, \sigma) \tag{6}$$

is defined over another pair of dual variables, namely (plastic strain rate, stress). In the case of standard materials, the plastic bipotential is separated (expressed as the sum of the plastic potential and it's dual). For implicit standard constitutive laws which can be expressed by a bipotential (like for example the non-associative Drücker-Prager law), the bipotential is not separated.

The constitutive equations are:

$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{7}$$

$$\varepsilon^e \in \partial b_e(\varepsilon^e, \cdot)(\sigma) \tag{8}$$

$$\dot{\varepsilon}^p \in \partial b_p(\dot{\varepsilon}^p, \cdot)(\sigma) \tag{9}$$

The constitutive equation (8) is equivalent with $\varepsilon^e = S^{-1}\sigma$, which is a linear equation. In order to enhance the resemblance between (8) and (9), we could differentiate with respect to time in the constitutive equation for ε^e and then express the result with the help of the elastic bipotential:

$$\dot{\varepsilon}^e \in \partial b_e(\dot{\varepsilon}^e, \cdot)(\dot{\sigma}) \tag{10}$$

5. Non-associated Drücker-Prager elasto-plasticity

An important example of an implicit standard material is provided by the non-associated Drücker-Prager constitutive law. Here we follow the presentation from [1].

5.1. **Plastically admissible stresses.** The model is characterized by a Drücker-Prager plastic yielding surface. The set of plastically admissible stresses is the following cone:

$$K_{stress} = \left\{ \sigma = \frac{1}{3} s_m I \, + \, s \text{ such that } \frac{1}{k_d}\|s\| + s_m \, tg\,\phi \leq c \right\}$$

Here c is the cohesion, ϕ is the friction angle and k_d is a constant whose significance is explained in [1] section 3, relations (3.1), (3.2).

We denote by $K'_{stress} = T(K_{stress})$ the same cone in coordinates (s_m, s) of the stresses.

5.2. **Plastically admissible strain rates.** Let $\theta \in [0, \phi]$ be the dilatancy angle (if $\theta = \phi$ then we are in the case of associated Drücker-Prager elastoplasticity). The set of admissible plastic strain rates is the cone:

$$K_{strain} = \left\{ \dot{\varepsilon}^p = \frac{1}{3}\dot{e}^p_m I \, + \, \dot{e}^p \text{ such that } k_d \, tg\,\theta \, \|\dot{e}^p\| \leq \dot{e}^p_m \right\}$$

We denote by $K'_{strain} = T(K_{strain})$ the same cone in the representation (e_m, e) of the strains.

5.3. **The flow rule.** The constitutive equation for the evolution of the plastic strain has the following expression:

$$\left(\left(\dot{e}^p_m + k_d(tg\,\phi \, - \, tg\,\theta)\|\dot{e}^p\| \right), \dot{e}^p \right) \, \in \, \partial \Psi_{K'_{stress}}(s_m, s) \tag{11}$$

Theorems 4.1, 4.2 from [1] are collected into the following.

Theorem 5.1. *Let* $b'_p : (\mathbb{R} \times Sym_0(n)) \times (\mathbb{R} \times Sym_0(n)) \to \mathbb{R} \cup \{+\infty\}$ *be the function:*

$$b'_p((e_m, e), (s_m, s)) = \begin{cases} C_1 e_m + C_2(s_m - \frac{c}{tg\,\phi})\|e\| & \text{if } (s_m, s) \in K'_{stress} \text{ and } \\ & (e_m, e) \in K'_{strain} \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

where $\|e\|$ *is the norm defined by* $\|e\|^2 = \langle e, e \rangle$ *and the constants*

$$C_1 = \frac{c}{tg\,\phi} \quad , \quad C_2 = k_d(tg\,\theta - tg\,\phi)$$

are coming from the flow rule (11). Then:

(a) b'_p *is a bipotential with respect to the duality product* $\langle \cdot, \cdot \rangle'$,

(b) *the non-associated Drücker-Prager constitutive equation for the evolution of the plastic strain (11) can be expressed with the help of the bipotential* b'_p *as*

$$b'_p((\dot{e}^p_m, \dot{e}^p), (s_m, s)) = \langle(\dot{e}^p_m, \dot{e}^p), (s_m, s)\rangle'$$

As an application of proposition 2, we obtain the following characterization of the Drücker-Prager constitutive law.

Corollary 1. *In the coordinates* $(\varepsilon, \sigma) \in Sym(n) \times Sym(n)$*, with the duality product* $\langle \varepsilon, \sigma\rangle = tr\,(\varepsilon\sigma)$*, the non-associated Drücker-Prager constitutive law (11) can be expressed with the help of the bipotential* $b_p : Sym(n) \times Sym(n) \to \mathbb{R} \cup \{+\infty\}$ *defined by:*

$$b_p(\varepsilon, \sigma) = \Psi_{K_{stress}}(\sigma) + \Psi_{K_{strain}}(\varepsilon) + C_1\, tr\, \varepsilon + C_2(tr\, \sigma - \frac{c}{tg\, \phi}) \| \varepsilon - \frac{1}{n}(tr\, \varepsilon) I \| - \left(1 - \frac{1}{n}\right)(tr\, \varepsilon)(tr\, \sigma) \tag{13}$$

Remark 2. The term containing C_2 represents a coupling between the hydrostatic part of the stress and the deviatoric part of the strain (rate). If $C_2 = 0$ then we get the associated Drücker-Prager constitutive law. In this case the last term from the right hand side of the expression (13) can be eliminated by modifying the cones K_{stress} and K_{strain}. But if $C_2 \neq 0$ such a modification cannot be made because of the coupling between deviatoric and hydrostatic parts, so this last term in the expression of b_p can not disappear by a modification of the cones K_{stress} and K_{strain}.

6. Time discretisation of the evolution problem

Given as the initial data the displacement u_0 and the initial plastic strain ε^p_0, the boundary data $\bar{u} = \bar{u}(t)$, $f_s = f_s(t)$, and the volume forces $f_v = f_v(t)$, for $t \in [0, T]$, a solution of the evolution problem is a collection $(u, \varepsilon^p, \varepsilon^e, \sigma)$ of fields dependent on t, which satisfy the kinematical, equilibrium, constitutive equations, as well as the initial and boundary conditions.

We want to give a variational formulation of the time discretisation of the evolution problem. For this we consider a discretisation

$$\{t_0 = 0, t_1, ..., t_N = T\}$$

of the time interval $[0, T]$. For each $k = 0, ..., N$ we denote by $(u_k, \varepsilon^p_k, \varepsilon^e_k, \sigma_k)$ the unknowns at the moment t_k. We shall use also the notation: for any $k = 0, ..., N$, let $\Delta t_k = t_{k+1} - t_k$, $\Delta u_k = u_{k+1} - u_k$, and so on, for all fields, known or unknown.

Further on, we shall replace the time derivatives from the evolution equation by finite differences with respect to the considered time discretisation. The problem which we want to solve is the following one.

6.1. **Problem (Pdisc).** Given $(u_k, \varepsilon^p_k, \varepsilon^e_k, \sigma_k)$, find $(\Delta u, \Delta\varepsilon^p, \Delta\varepsilon^e, \Delta\sigma)$, solution of the following problem:

$$\Delta\varepsilon^e + \Delta\varepsilon^p = D\,(\Delta u) \tag{14}$$

$$\Delta\varepsilon^e = S\Delta\sigma \tag{15}$$

$$\frac{1}{\Delta t_k}\Delta\varepsilon^p \in \partial b_p\left(\frac{1}{\Delta t_k}\Delta\varepsilon^p, \cdot\right)(\sigma_k + \Delta\sigma) \tag{16}$$

$$\Delta\sigma \in SA(\Delta f_{v,k}, \Delta f_{s,k}) \tag{17}$$

$$\Delta u \in CA(\Delta\bar{u}_k) \tag{18}$$

The unknowns $(u_{k+1}, \varepsilon^p_{k+1}, \varepsilon^e_{k+1}, \sigma_{k+1})$ are obtained as

$$u_{k+1} = u_k + \Delta u \quad , \quad \varepsilon^p_{k+1} = \varepsilon^p_k + \Delta\varepsilon^p \quad , ...$$

Our first concern is to express (Pdisc) with the help of bipotentials.

Lemma 6.1. *For any $k = 0, ..., N-1$, the function*

$$b_{p,k}(\Delta\varepsilon^p, \Delta\sigma) = \Delta t_k b_p\left(\frac{1}{\Delta t_k}\Delta\varepsilon^p, \sigma_k + \Delta\sigma\right) - \langle \Delta\varepsilon^p, \sigma_k\rangle \tag{19}$$

is a bipotential and the equation (16) is equivalent with

$$\Delta\varepsilon^p \in \partial b_{p,k}(\Delta\varepsilon^p, \cdot)(\Delta\sigma) \tag{20}$$

Proof. Let us show that

$$c_{p,k}(\Delta\varepsilon^p, \Delta\sigma) = b_{p,k}(\Delta\varepsilon^p, \Delta\sigma) - \langle \Delta\varepsilon^p, \Delta\sigma\rangle$$

is a sync. For this we introduce the sync associated to the bipotential b_p, namely

$$c_p(\Delta\varepsilon^p, \Delta\sigma) = b_p(\Delta\varepsilon^p, \Delta\sigma) - \langle \Delta\varepsilon^p, \Delta\sigma\rangle$$

Remark that

$$c_{p,k}(\Delta\varepsilon^p, \Delta\sigma) = \Delta t_k\, c_p\left(\frac{1}{\Delta t_k}\Delta\varepsilon^p, \sigma_k + \Delta\sigma\right)$$

We apply proposition 2 and get the result. By consequence, the function $b_{p,k}$ defined by (19) is a bipotential. From here, the second part of the proposition is a straightforward computation which is left for the interested reader. □

6.2. **Simplification of the boundary conditions and volume forces.** It is not a restriction of generality to suppose that the boundary conditions and volume forces are trivial, that is to suppose that equations (17), (18) have the following form:

$$\Delta\sigma \in SA(0, 0) \tag{21}$$

$$\Delta u \in CA(0) \tag{22}$$

Indeed, let us choose a field $\Delta\bar{\sigma} \in SA(\Delta f_{v,k}, \Delta f_{s,k})$. If we define the new unknowns:

$$\Delta u' = \Delta u - \Delta\bar{u} \quad , \quad \Delta\sigma' = \Delta\sigma - \Delta\bar{\sigma}$$

then we could use using again proposition 2 in order to prove that the constitutive equations, in the new unknowns, can be expressed by bipotentials.

In order not to use a too heavy notation, further on we shall assume (21), (22) and we shall neglect the change of unknowns (thus maintaining the notations Δu, $\Delta\sigma$).

6.3. **Elimination of several unknowns.** We can simplify the problem (Pdisc) by a standard argument involving the elimination of the unknowns $\Delta\varepsilon^e, \Delta\varepsilon^p$, by using an inf-convolution.

Indeed, let us denote $\Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^p$. By equation (14), $\Delta\varepsilon$ can be deduced from Δu.

For any $\Delta\sigma$, the functions $b_e(\cdot, \Delta\sigma)$ and $b_{p,k}(\cdot, \Delta\sigma)$ are not everywhere infinite, are convex and lower semicontinuous, therefore we can define the inf-convolution of them:

$$\Delta b_k(\Delta\varepsilon, \Delta\sigma) \,=\, \left(b_e(\cdot, \Delta\sigma) \,\square\, b_{p,k}(\cdot, \Delta\sigma)\right)(\Delta\varepsilon) \tag{23}$$

Lemma 6.2.

$$\Delta\sigma \in \partial\Delta b_k(\cdot, \Delta\sigma)(\Delta\varepsilon) \tag{24}$$

is equivalent with: there are $\Delta\varepsilon^e, \Delta\varepsilon^p$, such that $\Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^p$, which satisfy, together with $\Delta\sigma$, the equations (15), 16).

Proof. Indeed, by a well known property of inf-convolutions, equation (24) is equivalent with: there are $\Delta\varepsilon^e, \Delta\varepsilon^p$, such that $\Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^p$, which satisfy

$$\Delta\sigma \in \partial b_e(\cdot, \Delta\sigma)(\Delta\varepsilon^e) \tag{25}$$

$$\Delta\sigma \in \partial b_{p,k}(\cdot, \Delta\sigma)(\Delta\varepsilon^p) \tag{26}$$

But both b_e and $b_{p,k}$ are bipotentials, therefore (25) is equivalent with (15) and (26) is equivalent with (20), which is equivalent with (16) by lemma 6.1. □

Remark 3. Because of the particular form of b_e (quadratic function), the inf-convolution $\Delta b_k(\cdot, \Delta\sigma)$ is differentiable, with Lipschitz gradient, as a kind of Moreau-Yosida regularization. Therefore the inclusion (24) is equivalent with a standard equality, because the set from the right hand side contains only one element. This is an well known advantage of this elimination of unknowns in associated plasticity.

Let us list the properties of the function Δb_k:

- it is lower semicontinuous (even differentiable, with Lipschitz gradient in the first argument)
- Δb_k is defined via an inf-convolution of a bipotential of type (13) with the elastic bipotential b_e, therefore it satisfies the same growth inequality as b_e namely there is a constant $C > 0$ such that for any $\Delta\varepsilon \in Sym(n)$ and $\Delta\sigma \in Sym(n)$, if $\Delta b_k(\Delta\varepsilon, \Delta\sigma) < +\infty$ then
$$\Delta b_k(\Delta\varepsilon, \Delta\sigma) \,\leq\, C\,\left(\|\Delta\varepsilon\|^2 + \|\Delta\sigma\|^2\right)$$
where $\|\cdot\|$ is an arbitrary euclidean norm on the space $Sym(n)$,
- it satisfies a weak form of the Fenchel inequality,
$$\Delta b_k(\Delta\varepsilon, \Delta\sigma) \,\geq\, \langle\Delta\varepsilon, \Delta\sigma\rangle$$
- it is convex in the first argument, but not in the second, therefore it is not a bipotential, as it is stated in Theorem 6.1 [1]. Remark however that the proof of Lemma 6.2 uses the fact that the function $b_{p,k}$ is a bipotential.

We collect the partial results obtained so far into the following theorem, which provides a simplified form of the problem (Pdisc).

Theorem 6.3. *The problem (Pdisc) is equivalent with the following one: find* $(\Delta u, \Delta\sigma) \in CA(0) \times SA(0,0)$ *which satisfy (24).*

7. Variational formulation of the problem (Pdisc)

We give further a variational formulation à la Nayroles [15] of the following general problem, which contains (Pdisc) as a particular case.

We consider a first pair of spaces in duality:

- $X = L^2(\Omega, Sym(n))$ is the space of the deformation fields ε,
- $Y = L^2(\Omega, Sym(n))$ is the space of stress fields σ.

Instead of equalities $X, Y = L^2(\Omega, Sym(n))$, we may consider that X and Y are topological, locally convex, real vector spaces of dual variables $\varepsilon \in X$ and $\sigma \in Y$, with the duality product $\langle \cdot, \cdot \rangle_1 : X \times Y \to \mathbb{R}$, endowed with two injective continuous linear transformations $A : X \to L^2(\Omega, Sym(n))$ and $B : X \to L^2(\Omega, Sym(n))$ such that

$$\langle \varepsilon, \sigma \rangle_1 = \int_\Omega \langle A(\varepsilon)(x), B(\sigma)(x) \rangle \, \mathrm{d}x = \langle A(\varepsilon), B(\sigma) \rangle$$

In the integral we see the duality product (scalar product) on the space $Sym(n)$ of $n \times n$ symmetric real matrices. In the right hand side we see the duality product (scalar product) of L^2 with itself.

The space X, Y may be finite dimensional (for example associated with a discretisation in space by finite elements) or infinite dimensional. In the following we shall omit to mention the injections A, B or any other similar transformations which may appear. As an exception, in the following theorem 7.1, part (I), we need the spaces X, Y to be "large enough" in order to be able to prove that a solution of the variational formulation is also a solution (almost everywhere) of the original problem.

U is the space of $CA(0)$ displacement fields $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, with and $u = 0$ on $\partial_0\Omega$ in the sense of trace. The linear transformation $D : U \to X$, $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ is continuous and $V = D(U) \subset X$ is the image.

The space $Y_0 \subset Y$ of statically admissible $SA(0,0)$ stresses appear as the space of $\sigma \in Y$ with the property that for any $u \in U$ we have

$$\langle D(u), \sigma \rangle_1 = 0$$

We consider a function $b : Sym(n) \times K \to \mathbb{R}$ with the following properties:

(a) $K \subset Sym(n)$ is a closed convex set of the form $K = a + K_0$, with $a \in Sym(n)$ and $K_0 \subset Sym(n)$ a closed convex cone, such that $0 \in K$ (this is the set of plastically admissible stresses, as in the definition of the Drücker-Prager plasticity). Let $\pi_K : Sym(n) \to K$ be the projection on this cone;

(b) b is lower semicontinuous in both arguments, differentiable with Lipschitz gradient and convex in the first argument; moreover we suppose that the Lipschitz constant of the gradient of b in the first argument is continuous with respect to the second variable;

(c) b satisfies, for any $\varepsilon, \sigma \in Sym(n)$, the inequality: $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$,

(d) there is a constant $C > 0$ such that for any $\varepsilon \in Sym(n)$ and $\sigma \in K$ we have

$$b(\varepsilon, \sigma) \ \leq \ C \ \left(\|\varepsilon\|^2 + \|\sigma\|^2 \right) \tag{27}$$

Associated to the function b is the "bifunctional" of Berga and de Saxcé:

$$B(\varepsilon, \sigma) \ = \ \int_\Omega b(\varepsilon(x), \sigma(x)) \ \mathrm{d}x$$

Our main theorem is the following:

Theorem 7.1. *Suppose that the function b takes only finite values, that is for any $(\varepsilon, \sigma) \in Sym(n) \times Sym(n)$ we have $b(\varepsilon, \sigma) < +\infty$.*

(I) Let $u \in U$ and $\sigma \in Y_0$. The pair (u, σ) satisfies almost everywhere in Ω

$$\sigma \in \partial b(\cdot, \sigma)(D(u))$$

if and only if for any $\varepsilon \in X$ we have

$$B(D(u), \sigma) \ \leq \ B(\varepsilon, \sigma) - \langle \varepsilon, \sigma \rangle_1 \tag{28}$$

(II) For any $u^0 \in U$, $\sigma^0 \in Y_0$ there is a sequence $(u^k, \sigma^k)_k$ in $U \times Y_0$, such that for any $k \in \mathbb{N}$:

- *a. (global condition) for all $v \in U$ the displacement $u^{k+1} \in U$ satisfies*

$$B(D(u^{k+1}), \sigma^k) \leq B(D(v), \sigma^k)$$

- *b. (local condition) the stress σ^{k+1} satisfies almost everywhere in Ω the relation*

$$\sigma^{k+1} \in \partial b(\cdot, \sigma^k)(D(u^{k+1}))$$

(III) If a sequence $(u^k, \sigma^k)_k$ from (II) has a subsequence (denoted by same symbols) such that u^k converges weakly in $W^{1,2}$ to u and σ^k converges weakly in L^2 to σ, then (u, σ) is a solution of the problem (28).

Proof. (I) We follow the convention: we identify an element of $g \in L^2(\Omega, Sym(n))$ (which is an equivalence class of functions) with its representant, defined almost everywhere in Ω by Lebesgue theorem.

Let $u \in U$ and $\sigma \in Y_0$, such that we have $\sigma \in \partial b(\cdot, \sigma)(D(u))$ almost everywhere in Ω. Let us take $\varepsilon \in X$. Then, by integration of the constitutive relation (and by the definition of Y_0), we have

$$\int_\Omega b(\varepsilon(x), \sigma(x)) \ \mathrm{d}x \ - \int_\Omega \langle \varepsilon(x), \sigma(x) \rangle \ \mathrm{d}x \ \geq \int_\Omega b(D(u)(x), \sigma(x)) \ \mathrm{d}x$$

which is exactly the relation (28).

Conversely, let us start from the last integral inequality, supposed to be true for any $\varepsilon \in X$. Further we suppose also that $X = L^2(\Omega, Sym(n))$. Let us pick an arbitrary x_0 in the intersection of the Lebesgue sets of $D(u)$ and σ. For any open ball $B(x_0, r) \subset \Omega$ centered in x_0 we define $\varepsilon_r \in X$ such that $\varepsilon_r = D(u)$ almost everywhere outside B. We obviously get that

$$\frac{1}{\mid B(x_0, r) \mid} \left(\int_{B(x_0,r)} b(\varepsilon_r(x), \sigma(x)) \ \mathrm{d}x \ - \int_{B(x_0,r)} \langle \varepsilon_r(x) - D(u)(x), \sigma(x) \rangle \ \mathrm{d}x \right) \geq$$

$$\geq \ \frac{1}{\mid B(x_0, r) \mid} \int_{B(x_0,r)} b(D(u)(x), \sigma(x)) \ \mathrm{d}x \tag{29}$$

For any $\bar{\varepsilon} \in Sym(n)$ we can choose for any $r > 0$ (but sufficiently small) an ε_r such that

$$\lim_{r \to 0} \frac{1}{\mid B(x_0, r) \mid} \int_{B(x_0,r)} \varepsilon_r(x) \, \mathrm{d}x \, = \, \bar{\varepsilon}$$

and such that we can pass to the limit with r to 0, to obtain:

$$b(\bar{\varepsilon}, \sigma(x_0)) - \langle \bar{\varepsilon} - D(u)(x_0), \sigma(x_0) \rangle \, \geq \, b(D(u)(x_0), \sigma(x_0))$$

This is equivalent with the satisfaction of the constitutive relation almost everywhere.

(II) Suppose that for $k \in \mathbb{N}$ we defined the element (u^k, σ^k) of the sequence. We want to prove the existence of (u^{k+1}, σ^{k+1}) which satisfy the global condition (a), the local condition (b) and $\sigma^{k+1} \in Y_0$.

By the convexity, growth and continuity conditions on b, we easily obtain the existence of a minimizer of the functional $u \in U \; \mapsto \; B(D(u), \sigma^k)$. This proves the existence of u^{k+1} which satisfy the global condition (a). The local condition (b) is in fact the definition of σ^{k+1}. Because of the differentiability and continuity conditions on b, it easily follows from $\sigma^k \in Y$ that $\sigma^{k+1} \in Y$. We have to prove that $\sigma^{k+1} \in Y_0$. For this, we choose an arbitrary $v \in U$ and we integrate the local condition (b). We obtain that

$$B(D(v), \sigma^k) - B(D(u^{k+1}), \sigma^k) \, \geq \, \langle D(v) - D(u^{k+1}), \sigma^{k+1} \rangle$$

The left hand side of this inequality is non negative (by the global condition) and it can be made arbitrarily small, for example by choosing $v = u^{k+1} + \lambda w$, for a given, but arbitrary $w \in U$ and $\lambda > 0$ smaller and smaller. As a conclusion we obtain that for any $w \in U$ we have

$$\langle D(w), \sigma^{k+1} \rangle \leq 0$$

which implies that $\sigma^{k+1} \in Y_0$.

(III) Suppose that (u^k, σ^k) converges, in the given sense, to (u, σ). The sequence of functionals $v \mapsto B(D(v), \sigma^k)$ converges in the variational sense to the functional $v \mapsto B(D(v), \sigma)$, so, up to the choice of a subsequence, the minimizers of these respective functionals (namely the u^{k+1}) converge to a minimizer of the latter functional. Therefore (u, σ) satisfy the condition

$$B(D(v), \sigma) \geq B(D(u), \sigma)$$

for any $v \in U$.

The limit σ is in Y_0 by construction. We can also pass to the limit in the integral form of the local condition, which is: for any $\varepsilon \in X$

$$B(\varepsilon, \sigma^k) - \langle \varepsilon - D(u^{k+1}), \sigma^{k+1} \rangle \geq B(D(u^{k+1}), \sigma^k)$$

and we get the relation (28). □

The previous theorem contains at part (II) an algorithm for finding a solution of the problem (Pdisc). This algorithm is the rigorous reformulation of an algorithm proposed in [1] section 8.

However, this theorem can be improved (and will be, in further research) in several respects. Firstly, in the case of Drücker-Prager plasticity, the function Δb_k takes also infinite values. In this case the algorithm for solving the problem (Pdisc) should take the following form. Let K denote the set of plastically admissible stresses. Then:

0. initialize (u^0, σ^0) (for example take them equal to $(0,0)$,
1. repeat: given $(u^k, \sigma^k) \in U \times Y$,
   a. (global condition) find u^{k+1} such that for all $v \in U$
   $$B(D(u^{k+1}), \sigma^k) - \langle D(u^{k+1}), \sigma^k \rangle \leq B(D(v), \sigma^k) - \langle D(v), \sigma^k \rangle$$
   b. (local condition) define the stress σ^{k+1} almost everywhere in Ω by the relation
   $$\sigma^{k+1} \in \pi_K \left(\partial b(\cdot, \sigma^k)(D(u^{k+1})) \right)$$

We don't know yet how to prove that such a sequence converges to a solution of the problem (28), which is the weak form of problem (Pdisc).

Secondly, by exploiting the particular expression of the functions b which appear in real plasticity problems, we may be able to prove that sequences (u^k, σ^k) have convergent subsequences, for example by a boundedness argument.

Another, potentially very interesting subject, concerns Coulomb friction. This law can be expressed by a bipotential, [20] [7]. It should be interesting to explore the corresponding variational formulation, where the bifunctional will contain volume integrals as well as surface integrals. Related to this see also the paper [13].

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